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ON W^* EMBEDDING OF AW^*-ALGEBRAS[1]

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ABSTRACT. An AW^*-algebra N with a separating family of completely additive states and with a family $\{e_\alpha : \alpha \in A\}$ of mutually orthogonal projections such that $\text{lub}_\alpha e_\alpha = 1$ and $e_\alpha N e_\alpha$ is a W^*-algebra for each $\alpha \in A$ is shown to have a faithful representation as a ring of operators. This gives a new and considerably shorter proof that a semifinite AW^*-algebra with a separating family of completely additive states has a faithful representation as a ring of operators.

In [3], I. Kaplansky introduces a class of C^*-algebras which he calls AW^*-algebras. These algebras imitate rings of operators (W^*-algebras) algebraically and much of the structure of rings of operators holds for them. Although every ring of operators, regarded as an abstract C^*-algebra, is an AW^*-algebra, there exist AW^*-algebras which cannot be represented as rings of operators [**1**, Theorem 2]. In [**2**], J. Feldman shows that a finite AW^*-algebra with a separating family of states which are completely additive on projections can be represented as a ring of operators. He conjectures that the theorem is true without the assumption of finiteness. In [**5**], K. Saitô extends Feldman's result to semifinite AW^*-algebras. This paper presents a somewhat more general theorem which provides a considerably shorter independent proof of Saitô's result. In doing so it presents a technique which may be useful in attacking the still unsolved problem concerning a purely infinite AW^*-algebra with a separating family of completely additive states. For information on AW^*-algebras, see [**3**], and on W^*-algebras, see [**6**].

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DEFINITIONS. 1. A C^*-algebra is a uniformly closed *-algebra of operators on a Hilbert space.

2. An AW^*-algebra is a C^*-algebra satisfying:

(A) In the partially ordered set of projections, any set of orthogonal projections has a lub.

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(B) Any maximal commutative selfadjoint subalgebra is generated by its projections.

3. A ring of operators (or W^*-algebra) is a weakly closed *-algebra of operators on a Hilbert space.

4. An AW^* embedding of the AW^*-algebra N is a *-isomorphism ϕ of N into all bounded operators on some Hilbert space H such that the lub in $B(H)$ of any family of mutually orthogonal projections in $\phi(N)$ lies in $\phi(N)$.

REMARKS. 1. In an AW^*-algebra, the projections form a complete lattice [3].

2. In [2], J. Feldman shows that if N is an AW^*-algebra with a separating family of completely additive states $\{\omega:\omega \in \Omega\}$, then $\{\omega:\omega \in \Omega\}$ can be used to construct an AW^* embedding of N into all bounded operators on some Hilbert space. The construction is carried out in the usual manner.

We will be using the following theorem, which is due to K. Saitô [4, Theorem 1]. The proof given below is essentially that given in [4].

THEOREM 1. *Let M be a W^*-algebra and let φ be any positive linear functional in the predual of M. Let N be any set in S (the unit sphere of M) which is adherent to an element a in S in the s^*-topology. Then for any positive number ε and any projection e in M, there is a projection e_0 in M and a sequence $\{a_i:i=1, 2, \cdots\}\subset N$ such that $e_0 \leqq e$, $\varphi(e-e_0)<\varepsilon$ and* $\lim_{i\to\infty} \|a_i e_0 - a e_0\| = 0$.

PROOF. It is sufficient to prove the theorem for the case when $a=0$ and $e=1$. Then there is a net $\{a_\theta:\theta \in \Theta\}\subset N$ which is convergent to 0 in the s^*-topology and $b_\theta = a_\theta^* a_\theta$ converges to 0 in the s-topology with $\|b_\theta\| \leqq 1$. We write $b_\theta = \int_0^1 \lambda \, dE_{\theta,\lambda}$. Since $b_\theta \to 0$ strongly,

$$(\tfrac{1}{16})\varphi(1 - E_{\theta,1/4}) \leqq \int_{1/4}^1 \lambda^2 \, d\varphi(E_{\theta,\lambda}) \leqq \int_0^1 \lambda^2 \, d\varphi(E_{\theta,\lambda}) = \varphi(b_\theta^2) \to 0.$$

So $\varphi(1-E_{\theta,1/4})\to 0$. For θ_1 sufficiently large, we have $\varphi(1-E_{\theta,1/4})<\varepsilon/2$. Since

$$b_{\theta_1} E_{\theta_1,1/4} = \int_0^1 \lambda \, dE_{\theta_1,\lambda} E_{\theta_1,1/4} = \int_0^{1/4} \lambda \, dE_{\theta_1,\lambda},$$

we have $\|b_{\theta_1} E_{\theta_1,1/4}\| \leqq 1/4$. Writing $e_{\theta_1} = E_{\theta_1,1/4}$, we have $\|a_{\theta_1} e_{\theta_1}\| = \|e_{\theta_1} b_{\theta_1} e_{\theta_1}\|^{1/2} \leqq 1/2$. Consider the family $\{a_\theta e_{\theta_1}:\theta \geqq \theta_1\}$. Then $a_\theta e_{\theta_1} \to 0$ in the s^*-topology and $b'_\theta = e_{\theta_1} b_\theta e_{\theta_1}$ $(\in e_{\theta_1} M e_{\theta_1})$ converges to 0 in the s-topology. In the same way as above, we obtain a projection $e_{\theta_2} \leqq e_{\theta_1}$ such that $\varphi(e_{\theta_1} - e_{\theta_2}) < \varepsilon/4$ and $\|b'_{\theta_2} e_{\theta_2}\| \leqq 1/16$. Then $\|a_{\theta_2} e_{\theta_2}\| = \|e_{\theta_2} b_{\theta_2} e_{\theta_2}\|^{1/2} \leqq 1/4$. We can continue in this manner and obtain a decreasing sequence of projections $\{e_{\theta_i}: i=1, 2, \cdots\}$ in M and a sequence $\{a_i:i=1, 2, \cdots\}\subset N$ such that

$\varphi(e_{\theta_i}-e_{\theta_{i+1}})<\varepsilon/2^{i+1}$ ($e_{\theta_0}=1$) and $\|a_{\theta_i}e_{\theta_i}\|\leqq 1/2^i$. Letting $e_0=\text{glb}\, e_{\theta_i}$, we have $\varphi(1-e_0)=\sum_{i=0}^{\infty}\varphi(e_{\theta_i}-e_{\theta_{i+1}})<\varepsilon$ and $\|a_{\theta_i}e_0\|\leqq 1/2^i$ for all i. Thus $\lim_{i\to\infty}\|a_{\theta_i}e_0\|=0$.

The following theorem is the main result of the paper.

THEOREM 2. *Let N be an AW^*-algebra with a separating family $\{\omega:\omega\in\Omega\}$ of completely additive states. Suppose, further, N contains a mutually orthogonal family $\{e_\alpha:\alpha\in A\}$ of projections with* $\text{lub}_\alpha\, e_\alpha=1$ *and such that $e_\alpha Ne_\alpha$ is a W^*-algebra for each $\alpha\in A$. Then N can be represented as a ring of operators.*

PROOF. Since N has a separating family of completely additive states, N can be AW^* embedded as operators on a Hilbert space H. The family $\{\omega:\omega\in\Omega\}$ of states is used to construct H and the *-isomorphism ϕ of N into $B(H)$ (see Remark 2). We remark that the image of $e_\alpha Ne_\alpha$ under ϕ is weak operator closed in $B(H)$. Indeed, if we restrict each $\omega\in\Omega$ to $e_\alpha Ne_\alpha$, we obtain a separating family of normal states on the W^*-algebra $e_\alpha Ne_\alpha$. On the unit sphere $(e_\alpha Ne_\alpha)_S$ of $e_\alpha Ne_\alpha$, the topology induced by this family is equivalent to the topology induced by the predual of $e_\alpha Ne_\alpha$ and $(e_\alpha Ne_\alpha)_S$ is compact in the Ω-topology. Let b be any element in the weak operator closure of $\phi(e_\alpha Ne_\alpha)$ in $B(H)$. We can assume, without loss of generality, that $\|b\|=1$. By the Kaplansky Density Theorem, there is a net $\{\phi(b_\gamma):\gamma\in\Gamma\}$ contained in $\phi((e_\alpha Ne_\alpha)_S)$ converging weakly to b. It follows from the construction of ϕ and H that $\{b_\gamma:\gamma\in\Gamma\}$ converges to some $b_0\in(e_\alpha Ne_\alpha)_S$ in the Ω-topology. Since for any $c\in N$, the positive linear functional $\varphi_c(a)=\omega(c^*ac)$ is completely additive on projections [2], the restriction of φ_c to the W^*-algebra $e_\alpha Ne_\alpha$ is normal. Therefore $\varphi_c(b_\gamma)\to\varphi_c(b_0)$ for all $c\in N$ and thus $\phi(b_\gamma)\to\phi(b_0)$ in the weak operator topology in $B(H)$. Thus $b=\phi(b_0)\in\phi(e_\alpha Ne_\alpha)$. So we have $\phi(e_\alpha Ne_\alpha)$ is weakly closed in $B(H)$. Henceforth N will be used to denote both the original AW^*-algebra and its isomorphic image in $B(H)$. Let M be the weak closure of N in $B(H)$. M is a W^*-algebra. It will suffice to show that any projection in M lies in N.

Let p_0 be a projection in M. Let $\{p_\iota:\iota\in I\}$ be a maximal family of pairwise orthogonal projections in N with $p_\iota\leqq p_0$. Since N is AW^* embedded in $B(H)$, $\sum_\iota p_\iota$ (i.e. the least upper bound in M of finite sums) is in N. Suppose $p=p_0-\sum_\iota p_\iota\neq 0$. Then there is an α in A for which $pe_\alpha\neq 0$ and thus $\varphi(pe_\alpha)\neq 0$ for some normal state φ of M. By the Kaplansky Density Theorem, the unit sphere N_S of N is weakly dense in the unit sphere S of M. Thus N_S is s^* dense in S and p is an s^* limit point of N_S. By Theorem 1, for any $\varepsilon>0$ there is a nonzero projection $f\leqq e_\alpha$ and a sequence $\{a_i:i=1,2,\cdots\}\subset N$ such that $\varphi(e_\alpha-f)<\varepsilon$ and $\lim_{i\to\infty}\|a_if-pf\|=0$. Since $e_\alpha Ne_\alpha$ is weakly closed in $B(H)$, $e_\alpha Me_\alpha=e_\alpha Ne_\alpha\subset N$ and $f=e_\alpha fe_\alpha\in N$. So

$a_i f \in N$ for each i. Since N is uniformly closed, $pf \in N$. Now

$$|\varphi(pe_\alpha) - \varphi(pf)|^2 \leqq \varphi(p)\varphi(e_\alpha - f) < \varepsilon,$$

so that we may assume $pf \neq 0$. Then $0 \neq pfp \in N$ and thus the range projection of pfp is a nonzero subprojection of p which is in N. This contradicts the maximality of $\{p_\iota : \iota \in I\}$. Therefore $p_0 = \sum_\iota p_\iota \in N$. Thus every projection in M lies in N and $M = N$. This completes the proof.

REMARK. This theorem is true if, instead of a mutually orthogonal family of projections, we have an arbitrary family of projections $\{e_\alpha : \alpha \in A\}$ in N with $\text{lub}_\alpha\, e_\alpha = 1$ and such that $e_\alpha N e_\alpha$ is a W^*-algebra for each $\alpha \in A$.

COROLLARY. *Let N be a semifinite AW^*-algebra with a separating family of states which are completely additive on projections. Then N can be represented as a ring of operators.*

PROOF. Since N is semifinite, the unit contains a nonzero finite projection. Let $\{e_\alpha : \alpha \in A\}$ be a maximal family of orthogonal nonzero finite projections. If $\sum_\alpha e_\alpha \neq 1$, $e_0 = 1 - \sum_\alpha e_\alpha$ contains a nonzero finite projection, contradicting the maximality of $\{e_\alpha : \alpha \in A\}$. By **[2]**, $e_\alpha N e_\alpha$ is a W^*-algebra for each α. By the above theorem, N is a W^*-algebra.

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