PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 2, October 1972

ON THE CHARACTERIZATION OF ABELIAN W^*-ALGEBRAS

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ABSTRACT. In this note we present an elementary proof that an Abelian W^*-algebra is generated by the range of some real spectral measure.

Let H denote a separable Hilbert space and $\mathscr{B}(H)$ the algebra of all bounded operators on H. A subalgebra $\mathscr{A}$ of $\mathscr{B}(H)$ is called a W^*-algebra if $I \in \mathscr{A}$, if $T^* \in \mathscr{A}$ whenever $T \in \mathscr{A}$ and if $\mathscr{A}$ is closed in the weak operator topology.

Suppose E is a real compact spectral measure. Let $\mathscr{R}(E)$ denote the range of E and let $\mathscr{A}(E)$ be the W^*-algebra generated by $\mathscr{R}(E)$. (Note that $A \in \mathscr{A}(E)$ if and only if $A = \int_{-\infty}^{\infty} \phi(\lambda)\, dE(\lambda)$ where ϕ is an essentially bounded, complex-valued, borel measurable function on $\mathscr{R}$.) One obviously has that $\mathscr{A}(E)$ is an Abelian W^*-algebra. One of the fundamental theorems in the analysis of W^*-algebras states that all Abelian W^*-algebras arise in this manner (see Dixmier [1], Naimark [2], or Schwartz [3]). In this paper we present an elementary method for constructing a spectral measure whose range generates a given Abelian W^*-algebra.

Throughout this discussion, H will represent a fixed, separable Hilbert space. Let $\mathscr{L}$ be the lattice of all projections (selfadjoint) on H, and assume that $\mathscr{L}$ inherits the weak operator topology.

Let $\{e_1, e_2, \cdots\}$ be a countable dense subset of $\{x \in H \mid \|x\| = 1\}$. For each P in $\mathscr{L}$ set $\alpha(P) = \sum_{n=1}^{\infty} 2^{-n} \|P(e_n)\|^2$.

LEMMA 1. *The mapping* $\alpha: P \to \alpha(P)$ *is continuous from* $\mathscr{L}$ *into* [0,1] *and satisfies*

(i) $\alpha(0) = 0$ *and* $\alpha(I) = 1$,

(ii) *if* $P \geqq Q$ *then* $\alpha(P) \geqq \alpha(Q)$,

(iii) *if* $\alpha(P_n) \to \alpha(P)$ *and* $P_n \leqq P$ *for each* $n = 1, 2, \cdots$, *then* $P_n \to P$.

PROOF. One can easily check that α is continuous, $\alpha(0) = 0$ and $\alpha(I) = 1$.

Received by the editors October 18, 1971.

AMS 1969 *subject classifications.* Primary 4665, 4660.

Key words and phrases. Abelian W^*-algebras, spectral measure, lattice of projections, chain of projections.

[1] Partially supported by National Science Foundation Grants GP-28925 and GU-3171.

If $P \geqq Q$ then $\|P(x)\|^2 \geqq \|Q(x)\|^2$ for all x in H. Hence

$$\alpha(P) = \sum 2^{-n} \|P(e_n)\|^2 \geqq \sum 2^{-n} \|Q(e_n)\|^2 = \alpha(Q).$$

Suppose that $\alpha(P_n) \to \alpha(P)$ and $P_n \leqq P$ for each $n=1, 2, \cdots$. Then

$$\lim_n \sum_{i=1}^{\infty} 2^{-i}(\|P(e_i)\|^2 - \|P_n(e_i)\|^2) = 0.$$

But

$$\begin{aligned}\|P(e_i)\|^2 - \|P_n(e_i)\|^2 &= (Pe_i, e_i) - (P_n e_i, e_i) \\ &= ((P - P_n)e_i, e_i) = \|(P - P_n)e_i\|^2.\end{aligned}$$

Therefore $\|(P-P_n)e_i\| \to 0$ for each $i=1, 2, \cdots$, and hence $P_n \to P$ in $\mathscr{L}$.

LEMMA 2. *Let $\mathscr{C}$ be a chain in $\mathscr{L}$. There is a real spectral measure $E_{\mathscr{C}}$ with $\mathscr{C} \subset \mathscr{R}(E_{\mathscr{C}})$.*

PROOF. Let $\mathscr{C}'$ be the chain that results by adjoining (if necessary) 0 and I to $\mathscr{C}$. For each t in $[0, 1]$ let

$$E_t = \sup\{P \in \mathscr{C}' \mid \alpha(P) \leqq t\}.$$

Then $E_0=0$ and $E_1=I$. Also, for s, t in $[0, 1]$, $E_s E_t = E_{\min(s,t)}$. (This follows since α is order preserving.) By applying (iii) of Lemma 1, we have $E_t = \inf\{E_s | s > t\}$ for each t in $[0, 1)$. Therefore $\{E_t | 0 \leqq t \leqq 1\}$ is a resolution of the identity. We define the spectral measure $E_{\mathscr{C}}$ in the usual manner by setting $E_{\mathscr{C}}([s, t]) = E_t - E_s$ for $[s, t] \subset [0, 1]$ and then extending to the borel subsets of $[0, 1]$.

Note that for each P in $\mathscr{C}$, $E_{\mathscr{C}}([0, \alpha(P)]) = P$. Hence $\mathscr{C} \subset \mathscr{R}(E_{\mathscr{C}})$.

LEMMA 3. *Let $\mathscr{C}$ be the chain of projections $0 < P_1 < \cdots < P_n < I$, and suppose P is a projection that commutes with P_i for $1 \leqq i \leqq n$. Let $\mathscr{C}' = \{PP_i, P + (I-P)P_i, 0, I | 1 \leqq i \leqq n\}$. $\mathscr{C}'$ is a chain, and if $E_{\mathscr{C}}$, $E_{\mathscr{C}'}$ are the spectral measures associated to $\mathscr{C}$, $\mathscr{C}'$ respectively as in Lemma 2 then $\mathscr{R}(E_{\mathscr{C}}) \subset \mathscr{R}(E_{\mathscr{C}'})$ and $P \in \mathscr{R}(E_{\mathscr{C}'})$.*

PROOF. Each element of $\mathscr{C}'$ is obviously a projection, and one has

$$0 \leqq PP_1 \leqq \cdots \leqq PP_n \leqq P \leqq P + (I - P)P_1 \leqq \cdots \leqq P + (I - P)P_n \leqq I.$$

Hence, $\mathscr{C}'$ is a chain.

If $Q \in \mathscr{R}(E_{\mathscr{C}})$ then PQ, $(I - P)Q \in \mathscr{R}(E_{\mathscr{C}'})$. Hence $Q = PQ + (I-P)Q \in \mathscr{R}(E_{\mathscr{C}'})$. Also, $P = E_{\mathscr{C}'}([0, \alpha(P)]) \in \mathscr{R}(E_{\mathscr{C}'})$.

Let $P_1, P_2, \cdots$ be a fixed commuting family of projections on H. For each $n=1, 2, \cdots$, let $\mathscr{C}_n$ denote the chain obtained by starting with $0 < P_n < I$ and successively adjoining $P_{n-1}, P_{n-2}, \cdots, P_1$ as in Lemma 3. Let $E_{\mathscr{C}_n}$ be the spectral measure associated to $\mathscr{C}_n$ as Lemma in 2.

LEMMA 4. *For each $n=1, 2, \cdots, \{P_1, \cdots, P_n\} \subset \mathscr{R}(E_{\mathscr{C}_n})$ and $\mathscr{C}_n \subset \mathscr{C}_{n+1}$.*

PROOF. That $\{P_1, \cdots, P_n\} \subset \mathscr{C}_n$ follows by repeated applications of Lemma 3.

To see that $\mathscr{C}_n \subset \mathscr{C}_{n+1}$, observe that $\mathscr{C}_n$ is obtained by adjoining $P_{n-1}, \cdots, P_1$ to $\mathscr{C}: 0 < P_n < I$ and that $\mathscr{C}_{n+1}$ is obtained by adjoining $P_{n-1}, \cdots, P_1$ to $\mathscr{C}': 0 \leqq P_{n+1}P_n \leqq P_n \leqq P_n + (I - P_n)P_{n+1} \leqq I$. Since $\mathscr{C} \subset \mathscr{C}'$, the result follows immediately.

LEMMA 5. *Let $\mathscr{C} = \bigcup_{n=1}^{\infty} \mathscr{C}_n$. $\mathscr{C}$ is a chain and if $E_{\mathscr{C}}$ is the spectral measure associated to $\mathscr{C}$ as in Lemma 2, $P_n \in \mathscr{R}(E_{\mathscr{C}})$ for each $n = 1, 2, \cdots$.*

PROOF. Suppose $P, Q \in \mathscr{C}$; then there exist n, m such that $P \in \mathscr{C}_n$ and $Q \in \mathscr{C}_m$. If $N = \max\{n, m\}$, then $P, Q \in \mathscr{C}_N$. Hence, either $P = Q$, $P < Q$ or $P > Q$.

Since $P_n \in \mathscr{R}(E_{\mathscr{C}_n})$, there is a borel set M_n, determined by intervals with endpoints in $\{\alpha(P) | P \in \mathscr{C}_n\}$, such that $P_n = E_{\mathscr{C}_n}(M_n)$. But then, since $\mathscr{C}_n \subset \mathscr{C}$, $E_{\mathscr{C}}(M_n) = E_{\mathscr{C}_n}(M_n) = P_n$. Hence, $P_n \in \mathscr{R}(E_{\mathscr{C}})$ for each $n = 1, 2, \cdots$.

We can now easily prove

THEOREM 6. *Suppose $\mathscr{A}$ is a commutative selfadjoint subalgebra of $\mathscr{B}(H)$. There is a real spectral measure E such that the W^*-algebra generated by $\mathscr{R}(E)$ contains $\mathscr{A}$.*

PROOF. Let $\mathscr{A}_r$ be the hermitian elements of $\mathscr{A}$. For each A in $\mathscr{A}_r$, let E_A be the spectral measure such that $A = \int_{-\infty}^{\infty} \lambda \, dE_A(\lambda)$. Finally, let $\mathscr{P} = \bigcup_{A \in \mathscr{A}_r} \mathscr{R}(E_A)$, and let $\{P_1, P_2, \cdots\}$ be a dense subset of $\mathscr{P}$. If $\mathscr{C}$ and $E_{\mathscr{C}}$ are as in Lemma 5, with respect to $\{P_1, P_2, \cdots\}$, then clearly $\mathscr{A}$ is in the W^*-algebra generated by $\mathscr{R}(E_{\mathscr{C}})$.

REMARK. If the algebra $\mathscr{A}$ of Theorem 6 is a W^*-algebra, the set of projections $\mathscr{P}$ is contained in $\mathscr{A}$. Hence, the W^*-algebra generated by $\mathscr{R}(E)$ coincides with $\mathscr{A}$. It also follows in this case, that if $A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda)$, $\mathscr{A}$ is the W^*-algebra generated by A and I.

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