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QUADRATIC JORDAN ALGEBRAS WHOSE ELEMENTS ARE ALL INVERTIBLE OR NILPOTENT

KEVIN McCRIMMON

ABSTRACT. We prove that if $\mathfrak{J}$ is a unital quadratic Jordan algebra whose elements are all either invertible or nilpotent, then modulo the nil radical $\mathfrak{N}$ the algebra $\mathfrak{J}/\mathfrak{N}$ is either a division algebra or the Jordan algebra determined by a traceless quadratic form in characteristic 2. We also show that if $\mathfrak{A}$ is an associative algebra with involution whose symmetric elements are either invertible or nilpotent, then modulo its radical $\mathfrak{A}/\mathfrak{R}$ is a division algebra, a direct sum of anti-isomorphic division algebras, or a split quaternion algebra.

In [1] N. Jacobson filled a gap in A. A. Albert's structure theory for finite-dimensional Jordan algebras over a field Φ of characteristic $\neq 2$ by showing that if $\mathfrak{J}$ was almost nil (in the sense that every element could be written as $\alpha 1+z$ for $\alpha \in \Phi$ and z nilpotent) then the set $\mathfrak{N}$ of nilpotent elements formed an ideal (see also [2, p. 198]). This was extended by J. M. Osborn [9] to the case of an arbitrary Jordan algebra without 2-torsion in which every element was invertible or nilpotent. Such algebras occur naturally in the structure theory of Jordan algebras: D. L. Morgan has shown [8, Theorem 2.1] that if e is a primitive idempotent in a Jordan algebra $\mathfrak{J}$ with d.c.c. on inner ideals, then the Peirce subalgebra $\mathfrak{J}_1(e)$ has this property.

The result fails in characteristic 2—the Jordan algebras $J(Q, 1)$ of traceless quadratic forms Q satisfy $x^2=Q(x)1$ for all x, so either $Q(x)=0$ (and x is nilpotent) or $Q(x)\neq 0$ (and x is invertible). A particular case: $\mathfrak{J}$ consists of all matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix}$, so $x^2=(\det x)1$ for $\det x=\alpha^2+\beta\gamma$. However, in [4] it was shown that these were essentially the only counterexamples: any finite-dimensional almost nil quadratic Jordan algebra over a field Φ with more than two elements has nil radical $\mathfrak{N}$ such that $\mathfrak{J}/\mathfrak{N}$ is either $\Phi 1$ or $J(Q, 1)$ for a nondegenerate traceless Q. (We refer to [3] for all basic facts about quadratic Jordan algebras over an arbitrary ring of scalars Φ.)

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In the present paper we will extend this result to arbitrary quadratic Jordan algebras whose elements are either invertible or nilpotent, thus establishing Osborn's result for quadratic Jordan algebras.

THEOREM 1. *The quadratic Jordan algebras in which every element is either invertible or nilpotent are precisely those for which* $\mathfrak{J}/\mathfrak{N}$ ($\mathfrak{N}$ *the nil radical) is a division algebra or an algebra* $J(Q, 1)$ *determined by a traceless nondegenerate quadratic form* Q *over a field of characteristic* 2.

Since an element x of $\mathfrak{J}$ is nilpotent or invertible if and only if its image $\bar{x}$ in $\bar{\mathfrak{J}}=\mathfrak{J}/\mathfrak{N}$ is nilpotent or invertible, when $\mathfrak{J}/\mathfrak{N}$ has the stated form, the elements of $\mathfrak{J}$ are invertible or nilpotnet. We concentrate on showing the converse. If noninvertible elements are nilpotent the Jacobson radical Rad $\mathfrak{J}$ coincides with the nil radical $N(\mathfrak{J})$ (always $N(\mathfrak{J}) \subset$ Rad $\mathfrak{J}$, and Rad $\mathfrak{J}$ never contains invertible elements). Further, $\bar{\mathfrak{J}}$ shares the same property as $\mathfrak{J}$ but in addition is semisimple, so it suffices to prove

THEOREM 1′. *If* $\mathfrak{J}$ *is a semisimple quadratic Jordan algebra in which every element is invertible or nilpotent, then* $\mathfrak{J}$ *is either a division algebra or an algebra* $J(Q, 1)$ *determined by a traceless nondegenerate quadratic form over a field of characteristic* 2.

PROOF. If all nonzero elements are invertible, $\mathfrak{J}$ is a division algebra. Therefore we will always assume the set Z of nilpotent elements is not zero. Our goal is to find an isotope $\tilde{\mathfrak{J}}$ which has capacity two and use Osborn's Capacity Two Theorem to get $\tilde{\mathfrak{J}} \cong J(Q, 1)$.

Our first observation is that all z in Z square to zero, because in general

LEMMA 1. *If* $U_z a$ *is nilpotent for each* $a \in \mathfrak{J}$ *then* $z^2 \in$ Rad $\mathfrak{J}$.

PROOF. Since the radical Rad $\mathfrak{J}$ consists of the elements quasi-invertible in all homotopes $\mathfrak{J}^{(a)}$ [6], it suffices if z^2 is nilpotent in each $\mathfrak{J}^{(a)}$. Recall that an element is nilpotent if and only if its U-operator is nilpotent; but the U-operator of z^2 in $\mathfrak{J}^{(a)}$ is $U^{(a)}_{z^2}=U_{z^2}U_a=U_zU_zU_a$, which is nilpotent since ST is nilpotent iff TS is nilpotent, and $(U_zU_a)U_z=U_{U(z)a}$ is nilpotent by hypothesis. ■

Z cannot be an ideal if $\mathfrak{J}$ is semisimple, yet for any $z \in Z$ and $x \in \mathfrak{J}$, both $U_x z$ and $U_z x$ belong to Z (they cannot be invertible, since $U_x y$ is invertible iff x and y are, so they must be nilpotent), so the only way Z can fail to be an ideal is by not being closed under addition: $Z+Z \not\subset Z$. This means some sum of nilpotent elements z, w adds up to an invertible element $z+w=u$.

Already at this stage we can conclude $\mathfrak{J}$ has characteristic 2 ($2\mathfrak{J}=0$), and thus recapture Osborn's result. Indeed, if $z+w=u$ for $z^2=w^2=0$ then $1+w$ is invertible but $U_{1+w}z$ is nilpotent, so $0=U^{-1}_{1+w}\{U_{1+w}z\}^2=U^{-1}_{1+w}U_{1+w}U_z(1+w)^2=U_z(1+2w)=z^2+2U_zw=2U_zw=2U_uw$ (as $U_w w=w^3=0$, $U_{z,w}w=z \circ w^2=0$) and therefore $2w=0$ for $w\neq 0$. Since the centroid of our $\mathfrak{J}$ must be a field, this gives $2=0$. Instead of assuming characteristic 2, we prefer to keep our calculations general (the reader will see how they can be simplified if $2\mathfrak{J}=0$).

Once the elements of Z all square to zero, we can show that any z with $z+w=u$ invertible is von Neumann-regular and spawns an idempotent.

Lemma 2. *In any quadratic Jordan algebra, if $z+w=u$ is invertible for $z^2=z^3=w^2=0$ then $z=U_zy$, $y=U_yz$ for $y=U_{u^{-1}}z$. Whenever $z=U_zy$ for $z^2=y^2=0$ then $e=z\circ y$ is an idempotent with $U_ez=z$, hence $e\neq 0$ if $z\neq 0$.*

Proof. For the first assertion, $1=U_u^{-1}u^2=U_u^{-1}(z\circ w)=U_u^{-1}(z\circ u)$ (as $z\circ z=2z^2=0$) $=u^{-1}\circ z$, so

$$z = U_1z = U_{u^{-1}\circ z}z = \{U_{u^{-1}}U_z + U_zU_{u^{-1}} + U^2_{u^{-1},z} - U_{u^{-2},z^2}\}z = U_zU_{u^{-1}}z$$

since $U_zz=z^3=0$, $\{u^{-1}\,z\,z\}=u^{-1}\circ z^2=0$, $z^2=0$. Further,

$$U_yz = (U_{u^{-1}}U_zU_{u^{-1}})z = U_{u^{-1}}z = y.$$

For the second assertion, $e^2=(z\circ y)^2=U_zy^2+U_yz^2+y\circ U_zy=0+0+y\circ z=e$ is idempotent with

$$U_ez = U_{z\circ y}z = \{U_zU_y + U_yU_z + U^2_{z,y} - U_{y^2,z^2}\}z = U_zU_yz$$

(here $z^3=U_zz=U_zU_zy=U_{z^2}y=0$, $\{y\,z\,z\}=y\circ z^2=0$, $z^2=0)=U_zU_y(U_zy)=U_{U(z)y}y=U_zy=z$. ∎

In our situation the only idempotent other than zero is 1, so from $z+w=u$ we have found

$$z = U_zy, \qquad y = U_yz, \qquad z^3 = z^2 = y^2 = 0, \qquad z\circ y = 1.$$

The isotope we are looking for is $\tilde{\mathfrak{J}}=\mathfrak{J}^{(v)}$, $v=1+y+2z$. The element $v=1+w$ $(w=y+2z)$ is invertible with inverse $v^{-1}=-1+w$ $(=-1+y+2z)$ since $w^2=2y\circ z=2$. Here $e=z$ is an idempotent in $\tilde{\mathfrak{J}}$, since $e^{\tilde{2}}=U_ev=U_z(1+y+2z)=z^2+U_zy+2z^3=0+z+0=e$. The element $c=1-2z$ belongs to the Peirce space $\tilde{\mathfrak{J}}_{1/2}(e)$ since $e\,\tilde{\circ}\,c=\{z\,v\,1-2z\}=z\circ v-4U_zv=\{2z+z\circ y+4z^2\}-4z=1-2z=c$. Further, c squares to $\tilde{1}$ since $c^{\tilde{2}}=U_cv=U_c(c^{-1}+y)$ ($1-2z$ and $1+2z$ are inverses if $z^2=0$) $=c+U_{1-2z}y=(1-2z)+(y-2z\circ y+4U_zy)=1-2z+y-2+4z=-1+y+2z=v^{-1}=1^{(v)}=\tilde{1}$. The condition that noninvertible elements in $\mathfrak{J}=\{\mathfrak{J}^{(v)}\}^{(v^{-2})}=\tilde{\mathfrak{J}}^{(v^{-2})}$ square to zero becomes the condition that noninvertible elements in $\tilde{\mathfrak{J}}$ kill c,

$$x \text{ not invertible in } \tilde{\mathfrak{J}} \Rightarrow \tilde{U}_xc = 0$$

since $0=x^2=\tilde{U}_xv^{-2}$ where $v^{\tilde{-2}}=(w-1)^2=w^2-2w+1=3-2w=1-2v^{-1}=(1-2z)+2z-2v^{-1}=c+2e-2\tilde{1}=c-2f$ gives $\tilde{U}_x(c-2f)=0$; in particular, $\tilde{U}_f(c-2f)=-2f=0$, so that $\tilde{U}_xc=0$ for such x.

To apply the Capacity Two Theorem we need to know that $\tilde{\mathfrak{J}}_1$ and $\tilde{\mathfrak{J}}_0$ are division algebras and $\tilde{\mathfrak{J}}$ contains no trivial elements. But $\tilde{\mathfrak{J}}$ is semisimple since $\mathfrak{J}$ is, and a semisimple algebra contains no trivial elements [5], so we can apply

LEMMA 3. *Let $\mathfrak{J}$ be a quadratic Jordan algebra without trivial elements, e an idempotent, and $c \in \mathfrak{J}_{1/2}(e)$ an element with $c^2=1$ such that*

$$x \text{ not invertible in } \mathfrak{J} \text{ implies } U_x c = 0.$$

Then $\mathfrak{J}$ is semisimple of connected capacity 2, and the involution U_c is trivial on $\mathfrak{J}_{1/2}(e)$.

PROOF. Here U_c is an automorphism of $\mathfrak{J}$ of period 2 taking e into $f=1-e$; clearly c strongly connects e and f, and $\mathfrak{J}_0=U_c\mathfrak{J}_1$ will be a division algebra if $\mathfrak{J}_1$ is. All that remains is to show $\mathfrak{J}_1$ is a division algebra and U_c is trivial on $\mathfrak{J}_{1/2}$.

Suppose a is not invertible in $\mathfrak{J}_1$. Then a, f, and $a+f$ are not invertible in $\mathfrak{J}$, so by hypothesis $0=U_{a+f}c-U_ac-U_fc=\{a\,c\,f\}=a\circ(c\circ f)=a\circ c$. But if $a\circ c=0$ then a is trivial: $U_a\mathfrak{J}=U_a\mathfrak{J}_1=U_a(U_c\mathfrak{J}_0)=U_{a\circ c}\mathfrak{J}_0=0$. By hypothesis the only trivial element $\mathfrak{J}$ can have is zero, so the only element of $\mathfrak{J}_1$ which is not invertible is zero, showing $\mathfrak{J}_1$ to be a division algebra.

To show the involution U_c is the identity on $\mathfrak{J}_{1/2}$, take any element x in $\mathfrak{J}_{1/2}$ and consider $y=a_1+x+f$ (where $x^2=a_1+a_0$ for a_i in $\mathfrak{J}_i$). This modification of x is noninvertible since

$$\begin{aligned} U_y(e-x+a_0) &= \{U_{a_1}+U_x+U_f+U_{a_1,x}+U_{a_1,f}+U_{x,f}\}(e-x+a_0) \\ &= a_1^2+(U_xe-x^3+U_xa_0)+a_0+(a_1\circ x-a_1\circ x^2) \\ &\quad -a_1\circ x+(-x^2\circ f+x\circ a_0) \quad \text{(by the Peirce relations)} \\ &= a_1^2+a_0-x^3+a_1^2+a_0+x^3-2a_1^2-x^3-2a_0+x^3 \\ &= 0 \quad \text{(using the fact that } U_xa_i=a_j^2,\ a_i\circ x=x^3). \end{aligned}$$

Therefore by the assumed property of $\mathfrak{J}$ we have

$$U_yc=0,$$

or

$$\begin{aligned} 0 &= U_fU_yc = U_fU_yV_fc = \{-V_fU_yU_f+U_{U(f)y,V(f)y}\}c \\ &= -0+U_{f,x+2f}c = U_{f,x}c = U_f(x\circ c), \end{aligned}$$

the component of $x\circ c$ in $\mathfrak{J}_0$. Thus $U_cx=c\circ U_f(c\circ x)-x\circ U_e(c^2)=-x\circ e=-x$. But since $U_cc=c^3=c$ we have $c=-c$, $1=-1$, so also $U_cx=x$. ∎

Once $\tilde{\mathfrak{J}}$ is strongly connected semisimple of capacity two with trivial involution $\tilde{U}_c=I$ on $\tilde{\mathfrak{J}}_{1/2}$, we can apply Osborn's Capacity Two Theorem [3] to conclude $\tilde{\mathfrak{J}}\cong J(\tilde{Q},\tilde{1})$ for a nondegenerate quadratic form $\tilde{Q}$. Thus

$\mathfrak{J}\cong\mathfrak{J}(\tilde{Q}, \tilde{1})^{(v^{-2})}=J(Q, 1)$ is also determined by a nondegenerate quadratic form. The condition that each element be invertible ($Q(x)\neq 0$) or nilpotent ($Q(x)=T(x)=0$) then implies Q is traceless of characteristic 2. ∎

REMARK. Rather than prove that the involution is trivial, one can consider the other possibility in Osborn's Theorem: $\mathfrak{J}\cong\mathfrak{H}(\Delta_2, \Delta_0)$ with $e\cong\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $c\cong\begin{pmatrix}0&1\\1&0\end{pmatrix}$, for Δ an associative division algebra *with nontrivial involution* and ample subspace Δ_0. This possibility can be ruled out as follows: any $y=\begin{pmatrix}\delta\bar{\delta}&\delta\\\bar{\delta}&1\end{pmatrix}$ is noninvertible (it kills $\begin{pmatrix}1&-\delta\\-\bar{\delta}&\bar{\delta}\delta\end{pmatrix}$), so by hypothesis

$$0 = ycy = \begin{pmatrix}\delta\bar{\delta} & \delta\\ \bar{\delta} & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\delta\bar{\delta} & \delta\\ \bar{\delta} & 1\end{pmatrix} = \begin{pmatrix}\delta(\delta+\bar{\delta})\bar{\delta} & \delta(\delta+\bar{\delta})\\ (\delta+\bar{\delta})\bar{\delta} & \delta+\bar{\delta}\end{pmatrix}.$$

Thus $\delta+\bar{\delta}=0$ for all δ, so $1=-1$, $\bar{\delta}=\delta$ contradicts the nontriviality of the involution on Δ.

Osborn applied this result to $\mathfrak{J}=\mathfrak{H}(\mathfrak{A}, *)$ to show that if the symmetric elements of an associative algebra $\mathfrak{A}$ in characteristic $\neq 2$ are all invertible or nilpotent, then $\mathfrak{A}$ is a division algebra, direct sum of anti-isomorphic division algebras, or a split quaternion algebra. We will obtain the same result for arbitrary characteristics.

Recall that a subspace $\mathfrak{J}$ of symmetric elements of an algebra $\mathfrak{A}$ is *ample* if it contains 1 and $x\mathfrak{J}x^*\subset\mathfrak{J}$ for all x in $\mathfrak{A}$ (therefore it contains all norms xx^* and all traces $x+x^*$). If $\frac{1}{2}\in\Phi$ then the only ample subspace of $\mathfrak{H}(\mathfrak{A}, *)$ is $\mathfrak{H}(\mathfrak{A}, *)$ itself. An ample subspace is automatically a Jordan subalgebra.

THEOREM 2. *If $\mathfrak{A}$ is a unital associative algebra with involution and $\mathfrak{J}$ an ample subspace of $\mathfrak{H}(\mathfrak{A}, *)$ such that each element of $\mathfrak{J}$ is invertible or nilpotent, then $\mathfrak{A}/\mathrm{Rad}\,\mathfrak{A}$ is either*: (i) *an associative division algebra Δ with involution*, (ii) *a direct sum $\Delta\oplus\Delta^\circ$ of an associative division algebra with its opposite under the exchange involution*, (iii) *a split quaternion algebra Φ_2 over its center Φ with standard involution.*

PROOF. $\bar{\mathfrak{A}}=\mathfrak{A}/\mathrm{Rad}\,\mathfrak{A}$ is semisimple with involution, and $\bar{\mathfrak{J}}\subset\mathfrak{H}(\bar{\mathfrak{A}}, *)$ is still ample (but note that if $\mathfrak{J}=\mathfrak{H}(\mathfrak{A}*)$, it still may happen in characteristic 2 that $\bar{\mathfrak{J}}=\overline{\mathfrak{H}(\mathfrak{A}, *)}$ is not all of $\mathfrak{H}(\bar{\mathfrak{A}}, \bar{*})$, so we are forced to consider the case of a general ample subspace). Furthermore, $\bar{\mathfrak{J}}$ is semisimple as a Jordan subalgebra [7]. It therefore suffices to prove the result for $\bar{\mathfrak{A}}$ in place of $\mathfrak{A}$, i.e. *we assume $\mathfrak{A}$ and $\mathfrak{J}$ are semisimple.*

In this case $\mathfrak{A}$ is $*$-simple: if $\mathfrak{B}$ were a proper $*$-ideal, then for each b in $\mathfrak{B}$ the elements $(1-b)(1-b^*)=1-z$ and $(1-b^*)(1-b)=1-w$ are invertible (since $z=b+b^*-bb^*$ and $w=b+b^*-b^*b$ lie in $\mathfrak{B}\cap\mathfrak{J}$, and cannot be invertible if $\mathfrak{B}$ is proper, so must be nilpotent); thus $1-b$ is invertible, each b is quasi-invertible, and $\mathfrak{B}\subset\mathrm{Rad}\,\mathfrak{A}=0$.

If the elements of $\mathfrak{J}$ are all invertible then by the Herstein-Kleinfeld-Osborn Theorem [3], $\mathfrak{A}$ is Δ, $\Delta\oplus\Delta^\circ$, or Φ_2. If the elements of $\mathfrak{J}$ are not all invertible then, by Theorem 1, $x^2=Q(x)1$ for all x in $\mathfrak{J}\cong J(Q, 1)$, and we can go back to the situation where z, y in $\mathfrak{J}$ satisfy $z^2=y^2=0$, $z\circ y=1$. Then $zy+yz=1$, $z=zyz$, $y=yzy$ so the elements $e_{11}=zy$, $e_{22}=yz$, $e_{12}=z$, $e_{21}=y$ are a family of matrix units. This gives us $\mathfrak{A}\cong\mathfrak{D}_2$ for $\mathfrak{D}=e_{11}\mathfrak{A}e_{11}$ under the correspondence $x=a+be_{12}+e_{21}c+e_{21}de_{12}\leftrightarrow\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$. Here $\mathfrak{J}$ contains $z\cong e_{12}$ and $y\cong e_{21}$. $\mathfrak{D}$ inherits an involution $a\to\bar{a}=e_{12}a^*e_{21}$ (because $e_{12}^*=e_{12}$, $e_{21}^*=e_{21}$, $e_{11}^*=e_{22}$ if $z^*=z$, $y^*=y$), relative to which the involution $x\to x^*$ in $\mathfrak{A}$ corresponds to

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\to\begin{pmatrix} \bar{d} & \bar{b}\\ \bar{c} & \bar{a}\end{pmatrix}$$

in $\mathfrak{D}_2$. Here $\mathfrak{H}(\mathfrak{D}_2, *)$ consists of the matrices $\left(\begin{smallmatrix} a & h\\ k & \bar{a}\end{smallmatrix}\right)$ for $a\in\mathfrak{D}$, h, $k\in\mathfrak{H}(\mathfrak{D}, -)$.

The condition $x^2=Q(x)1$ implies $x^2\circ y=0$ for any x, y in $\mathfrak{J}$ or $x^2y=yx^2$ (characteristic 2!!). Observe that $\left(\begin{smallmatrix} d & 0\\ 0 & \bar{d}\end{smallmatrix}\right)$ belongs to $\mathfrak{J}$ since it is the trace of $\left(\begin{smallmatrix} d & 0\\ 0 & 0\end{smallmatrix}\right)$ for any d in $\mathfrak{D}$. Thus

$$\begin{pmatrix} a & 0\\ 0 & \bar{a}\end{pmatrix}^2=\begin{pmatrix} a^2 & 0\\ 0 & \bar{a}^2\end{pmatrix}$$

commutes with $\left(\begin{smallmatrix} b & 0\\ 0 & \bar{b}\end{smallmatrix}\right)$ for any a, b in $\mathfrak{D}$, and also with $z=\left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)$, so $a^2b=ba^2$ and $a^2=\bar{a}^2$, which shows that any a^2 lies in the subfield $\Phi=\mathfrak{H}\cap\Gamma$ of symmetric elements of the center Γ of $\mathfrak{D}$. At the same time, any $\left(\begin{smallmatrix} a\bar{a} & 0\\ 0 & a\bar{a}\end{smallmatrix}\right)$ belongs to $\mathfrak{J}$ as the norm of $\left(\begin{smallmatrix} 0 & a\\ a & 0\end{smallmatrix}\right)$ so it commutes with all the $\left(\begin{smallmatrix} b & 0\\ 0 & \bar{b}\end{smallmatrix}\right)$ for a, b in $\mathfrak{D}$, so $(a\bar{a})b=b(a\bar{a})$ and all norms lie in the center of $\mathfrak{D}$. Therefore $\mathfrak{D}$ is a composition algebra over Φ with the additional property that $a^2\in\Phi$ for all a in $\mathfrak{D}$; the only composition algebra with this property is $\mathfrak{D}=\Phi$, and our original algebra was a split quaternion algebra $\mathfrak{A}=\Phi_2$ with standard involution

$$\begin{pmatrix} \alpha & \beta\\ \gamma & \delta\end{pmatrix}\to\begin{pmatrix} \delta & \beta\\ \gamma & \alpha\end{pmatrix}=\begin{pmatrix} \delta & -\beta\\ -\gamma & \alpha\end{pmatrix}$$

(characteristic 2!!). ■

Note. Professors I. N. Herstein and M. S. Montgomery have recently improved Osborn's Theorem in several directions [10]. One of their results is

THEOREM 3. *If $\mathfrak{A}$ is an associative algebra with involution * such that each trace $x+x^*$ is either invertible or nilpotent, then $\mathfrak{A}/\mathrm{Rad}\,\mathfrak{A}$ is one of* (1) *a division ring Δ,* (2) *a direct sum $\Delta\oplus\Delta^\circ$ of a division ring with its opposite,*

(3) *a split quaternion algebra* Φ_2 *over its center, or* (4) *a commutative algebra of characteristic* 2 *with trivial involution.*

The generalization consists in repacing the set of all symmetric elements by the set of traces. (As usual, this makes a difference only when $\frac{1}{2} \notin \Phi$.) The set $\mathfrak{T}(\mathfrak{A}, *)$ of traces is not necessarily an ample subspace of $\mathfrak{H}(\mathfrak{A}, *)$. We always have $x\mathfrak{T}x^* \subset \mathfrak{T}$ since $xt(y)x^* = x(y+y^*)x^* = t(xyx^*)$. However, we need not have $1 \in \mathfrak{T}$ (and consequently $\mathfrak{T}$ need not contain all norms xx^*). In particular, $\mathfrak{T}$ need not be closed under squaring and so cannot be considered as a Jordan algebra $\mathfrak{J}$. Thus our results cannot be applied directly to this situation.

However, there is a way to reduce the case of traces to the case of norms. The idea is that whenever 1 is a trace, $y+y^*=1$, then $\mathfrak{T}$ becomes an ample subspace (and contains all norms $xx^* = xt(y)x^* = t(xyx^*)$), so our result applies. If the unit of $\mathfrak{A}$ is not a trace, we pass to an isotope $\mathfrak{A}^{(u)}$ whose unit u^{-1} *is* a trace; if the conclusions of Theorem 3 hold for $\mathfrak{A}^{(u)}$ they also hold for $\mathfrak{A}$.

To be more explicit, let us assume all traces $\mathfrak{T}(\mathfrak{A}, *)$ are invertible or nilpotent, where $\mathfrak{A}$ and $\mathfrak{H}(\mathfrak{A}, *)$ are semisimple. If all traces are zero, $x^* = -x$, then $\mathfrak{A}$ is commutative of characteristic 2 with trivial involution, and we have (4) of the theorem. If there are nonzero traces they cannot *all* be nilpotent, because of the general

LEMMA 4. *If z is an element of a Jordan algebra $\mathfrak{J}$ such that $z^2=0$ and $z \circ x$ is nilpotent, then z is nilpotent in $\mathfrak{J}^{(x)}$. If this happens for all x in $\mathfrak{J}$ then z belongs to* Rad $\mathfrak{J}$.

PROOF. If $z^2=0$ then z^3 is trivial ($U_{z^3}=0$), so we may assume $z^3=0$ too (for example, by dividing out the lower radical [5]). Then $U_z^{(x)} = U_z U_x$ is nilpotent because $(U_z U_x) U_z = U_{x \circ z} U_z$ implies $(U_z U_x)^{n+1} = U_{z\circ x}^n U_z U_x$ (note $U_{z \circ x} = U_z U_x + U_x U_z + U_{z,x}^2 - U_{z^2, x^2}$ where $U_z U_z = U_{z^2} = 0$ and $U_{z,x} U_z = U_{z^2,x} V_z - U_{z^3,x} = 0$ if $z^2 = z^3 = 0$). Consequently z is nilpotent in the homotope $\mathfrak{J}^{(x)}$. If z is quasi-invertible in all $\mathfrak{J}^{(x)}$ then it belongs to Rad $\mathfrak{J}$ by [6]. ■

The lemma applies to our $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, *)$ since if z is a trace so are all $zx + xz$ for x in $\mathfrak{H}(\mathfrak{A}, *)$. From this we conclude that if a trace z is nilpotent it squares to zero: the associative $*$-algebra $\mathfrak{B} = z\mathfrak{A}z$ has all its traces nilpotent (no $t(zaz) = zt(a)z$ can be invertible), so by the above is a radical algebra; then all $(za)z$ are quasi-invertible, so all z^2a are too, and z^2 lies in Rad $\mathfrak{A} = 0$.

Therefore some trace $x + x^* = u^{-1}$ is invertible. The isotope $\mathfrak{A}^{(u)}$ (with multiplication $x \cdot_u y = xuy$) has unit $1^{(u)} = u^{-1}$ a trace. Since invertible elements of $\mathfrak{A}$ are invertible in $\mathfrak{A}^{(u)}$, if nilpotent traces in $\mathfrak{A}$ stay nilpotent in $\mathfrak{A}^{(u)}$ then Theorem 2 will apply to $\mathfrak{A}^{(u)}$ to give (1), (2), or (3).

On the other hand, if some nilpotent trace z does not stay nilpotent in $\mathfrak{A}^{(u)}$ we claim 1 is already a trace in $\mathfrak{A}$, so Theorem 2 applies to $\mathfrak{A}$ itself. Indeed, since $z^2=0$ but z is not nilpotent in $\mathfrak{A}^{(u)}$, by Lemma 4 the trace $z \circ u$ must not be nilpotent in $\mathfrak{A}$, consequently $zu+uz=v$ is invertible. Now uzu is a trace which is not invertible, so it squares to zero, and $zu^2z=u^{-1}(uzu)^2u^{-1}=0$. Note $zv=zuz=vz$ since $z^2=0$. Then $v^2=(zu+uz)^2=zuzu+uzuz=vzu+uzv$, so $1=v^{-1}(v^2)v^{-1}=zuv^{-1}+v^{-1}uz$ is a trace (u, v, z being symmetric). ■

An even better trick is due to Professor Tamagawa. Instead of keeping the involution the same but changing the algebra to $\mathfrak{A}^{(u)}$, he keeps the algebra the same but changes the involution to $x^{*(u)}=ux^*u^{-1}$. Here $T(\mathfrak{A}, *(u))=uT(\mathfrak{A}, *)$ since $x+x^{*(u)}=x+ux^*u^{-1}=u[(u^{-1}x)+(u^{-1}x)^*]$, so if $u^{-1} \in T(\mathfrak{A}, *)$ then $1 \in T(\mathfrak{A}, *(u))$. The problem of showing the elements of $uT(\mathfrak{A}, *)$ are invertible or nilpotent in $\mathfrak{A}$ is equivalent to showing the elements of $T(\mathfrak{A}, *)$ are invertible or nilpotent in $\mathfrak{A}^{(u)}$.

We remark that the trick of passing to an isotope is not available where traces are merely regular or nilpotent.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22904