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ON THE ISOMORPHISMS BETWEEN CERTAIN CONGRUENCE GROUPS

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ABSTRACT. In this paper we study the possible isomorphisms between the congruence subgroups of the classical groups over integral domains by applying the involution-free techniques previously used by O'Meara and the author. We prove, in dimensions at least 6 and characteristic not 2, that a linear congruence group is never isomorphic to a symplectic congruence group nor to an isotropic unitary congruence group whose associated hermitian space has Witt index at least 3.

Introduction. Let W be an m-dimensional vector space over the field F, $m\geqq 3$. Let $^{-}$ denote the natural homomorphism of $\mathrm{GL}(W)$ onto $\mathrm{PGL}(W)$. For any subgroup S of $\mathrm{GL}(W)$, we define the projective group of S (which we denote PS) to be the image of S under the $^{-}$ homomorphism of $\mathrm{GL}(W)$ onto $\mathrm{PGL}(W)$.

Let V be an n-dimensional vector space over a field F_1 of characteristic not 2, and let J be an involutionary automorphism of F_1. Let (x, y) be a nondegenerate skew-hermitian form on V with respect to J. That is

$$(\lambda x + y, z) = \lambda(x, z) + (y, z) \quad \text{and} \quad (x, y) = -(y, x)^J$$

for all $\lambda \in F_1$, $x, y, z \in V$. Define

$$U_n(V) = \{\sigma \in \mathrm{GL}(V) \mid (\sigma x, \sigma y) = (x, y)\, \forall x, y \in V\}.$$

Thus $U_n(V)$ is the n-dimensional symplectic group when J is the identity map of F_1; otherwise $U_n(V)$ is a unitary group. *We will assume* (x, y) *has index at least* 3. For any subspace U of V we define the orthogonal complement U^* of U as $U^*=\{x \in V \mid (x, U)=0\}$. The radical of U, written rad U, is defined as rad $U=U\cap U^*$. U is called regular if rad $U=0$, U is degenerate if rad $U\neq 0$, and U is totally degenerate if rad $U=U$. Finally we say U is isotropic if there is a nonzero vector x in U with $(x, x)=0$. A regular two-dimensional isotropic subspace of V is called a hyperbolic plane.

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Recall that an invertible linear transformation τ that fixes pointwise all vectors in some hyperplane is called a shearing; if det $\tau=1$, τ is a transvection. If det $\tau\neq 1$, τ is a dilation. If τ is a shearing and $\tau\neq 1$ then the image of $\tau-1$ is a line, called the proper line of τ, and the hyperplane of fixed vectors of τ is called the proper hyperplane of τ. Now we say a subgroup G of PGL(W) is full of projective transvections if, for each line L and hyperplane H of W with $L\subset H$, there is a nontrivial transvection τ in GL(W) with proper spaces $L\subset H$ such that $\bar{\tau}\in G$. Similarly we say a subgroup G_1 of $PU(V)$ has enough transvections if for each isotropic line L of V there is a nontrivial transvection τ in $U(V)$ with proper line L such that $\bar{\tau}\in G_1$. In all that follows we will assume G and G_1 are two subgroups of PGL(W) and $PU(V)$ respectively such that G is full of transvections and G_1 has enough transvections.

In all that follows whenever we refer to G_1, G, V, W, F_1, F, n or m, we shall understand them to be defined as above. And we put:

$$\Delta=\{\sigma\in \mathrm{GL}_m(W)\mid \bar{\sigma}\in G\},\qquad \Delta_1=\{\sigma\in U_n(V)\mid \bar{\sigma}\in G_1\}.$$

If ρ is a linear functional on W and $a\in W$ we let $\tau_{a,\rho}$ denote the transvection defined by $\tau_{a,\rho}(x)=x+\rho(x)\cdot a$ for all $x\in W$. Similarly if a is an isotropic vector in V and λ is an element of F_1 such that $\lambda=\lambda^J$ then $\tau_{a,\lambda}$ denotes the transvection in $U_n(V)$ defined by $\tau_{a,\lambda}(x)=x+\lambda(x,a)a$. If $X\subseteq\Delta$ or $X\subseteq\Delta_1$, $C(X)$ denotes the centralizer $C_\Delta(X)$ or $C_{\Delta_1}(X)$ respectively. If $X\subseteq G$ or $X\subseteq G_1$, $C(X)$ denotes the centralizer $C_G(X)$ or $C_{G_1}(X)$ respectively; and if X is any group, DX is the commutator subgroup. For a line L and a hyperplane H of W with $L\subset H$ we let $\bar{T}(L,H)$ stand for the group of all projective transvections in G having proper line L and proper hyperplane H.

We say two elements σ_1 and σ_2 of GL(W) anticommute if $\bar{\sigma}_1$ and $\bar{\sigma}_2$ do commute, but σ_1 and σ_2 do not; i.e. $\sigma_1\sigma_2=\lambda\sigma_2\sigma_1$ for some scalar λ in F unequal to 0 or 1. And if a, b are any two elements of a group, $[a,b]$ denotes the commutator $aba^{-1}b^{-1}$.

If σ is an element of GL(W) we will call $(\sigma-1)W$ the residual space R of σ. In what follows the letter R will always denote the residual space of σ; similarly for R_i and σ_i. And for a subspace U of W we define $E(U)=\{\sigma\in\Delta\mid R\subseteq U\}$ where R is the residual space of σ. Similarly if U is a subspace of V, $E(U)$ will denote $\{\sigma\in\Delta_1\mid R\subseteq U\}$ where again R is the residual space of σ.

1.1. *Suppose that $\Sigma\in U_n(V)$ is such that $\Sigma(F_1a)\neq F_1a$ for some isotropic line F_1a of V. Let $\tau_{a,\lambda}$ be a nontrivial transvection in $U_n(V)$. Then Σ and $\tau_{a,\lambda}\Sigma^{-1}\tau_{a,-\lambda}$ do not commute.*

PROOF. Suppose they did commute; then $\tau_{\Sigma a,\lambda}\tau_{a,-\lambda}=\tau_{a,\lambda}\tau_{\Sigma^{-1}a,-\lambda}$. Choose a vector x orthogonal to $\Sigma^{-1}a$ but not to a. Upon substitution of x in the above equation we obtain

$$-\lambda(x, a)a + \lambda(x, \Sigma a)\Sigma a - \lambda^2(a, \Sigma a)(x, a)\Sigma a = \lambda(x, a)a$$

which implies $a \in F \cdot \Sigma a$, a contradiction. Q.E.D.

1.2. *Let $\sigma \in \Delta_1$ be such that* $\dim R=2$ *and $\sigma|R$ is not a scalar. Then $E(R)\subseteq CDC(\sigma)$.*

PROOF. As in 2.1 of [9]. Q.E.D.

1.3. *Let $\sigma \in \Delta_1$ and suppose* $\dim R\leqq 2$. *Then $CDC(\bar\sigma)\subseteq (E(R))^-$.*

PROOF. As in 2.2 of [9]. Q.E.D.

1.4. *Let $\sigma \in \Delta_1$ be such that R is a hyperbolic plane, and $\sigma|R$ is not a scalar. Then $CDC(\bar\sigma)=(E(R))^-$.*

PROOF. Apply 1.2 and 1.3. Q.E.D.

1.5. *Let $\sigma \in \Delta_1$ be such that* $\dim R\leqq 2$. (a) *If R is totally isotropic, $CDC(\bar\sigma)$ is abelian.* (b) *If R is a nonisotropic line, $CDC(\bar\sigma)$ is abelian.* (c) *If R is a hyperbolic plane and $\sigma|R$ is not a scalar, then $CDC(\bar\sigma)$ is non-abelian.*

PROOF. (a) follows from 1.3 and the fact if σ_1 and σ_2 are in $U_n(V)$ and $R_1\subseteq R_2^*$ then σ_1 and σ_2 commute.

(b) follows from 1.3.

To prove (c) observe that 1.2 implies $(E(R))^-\subseteq CDC(\bar\sigma)$. Since R is a hyperbolic plane there are two nonorthogonal isotropic lines L_1 and L_2 in R. If we choose two projective transvections in G whose proper lines are L_1 and L_2, then these two projective transvections will be in $(E(R))^-$, hence in $CDC(\bar\sigma)$, but will not commute. Q.E.D.

2.1. *Let W be an m-dimensional vector space over a field F of any characteristic, $m\geqq 4$. Let G be a subgroup of* $\mathrm{PGL}_m(W)$ *which is full of projective transvections and let $\bar\sigma \in G$ be a nontrivial projective transvection with proper spaces $L\subseteq H$. Then $CDC(\bar\sigma)\subseteq \bar T(L, H)$.*

PROOF. To begin, we claim that for any line L_1 in H there is a nontrivial transvection τ in $DC(\sigma)$ with proper line L_1.

For let $L_1=F\cdot x_1$, let $L=F\cdot x_0$ and if $L_1\neq L$ extend x_0 and x_1 to a basis $x_0, x_1, \cdots, x_{m-1}$ for W with $x_0, x_1, \cdots, x_{m-2}$ in H. Since Δ is full of transvections, there are nonzero α, β in F such that $\tau_{\alpha x_1,\rho_2}$ and $\tau_{x_2,\beta\rho_3}$ are in Δ where (ρ_i) is the dual base of (x_i). Clearly $\tau_{\alpha x_1,\rho_2}$ and $\tau_{x_2,\beta\rho_3}$ are in $C(\sigma)$ by

1.6 of [5], so $\tau=[\tau_{\alpha x_1,\rho_2};\tau_{x_2,\beta\rho_3}]=\tau_{\alpha x_1,\beta\rho_3}$ is a nontrivial transvection in $DC(\sigma)$ with proper line L_1.

Now if $L_1=L$ let $L_1=F\cdot x_1$ and extend x_1 to a base $x_1,\cdots,x_m$ for W with $x_1,\cdots,x_{m-1}$ in H. Then reasoning as above we will obtain a nontrivial transvection in $DC(\sigma)$ with proper line L_1.

Now let $\bar{\Sigma}\in CDC(\bar{\sigma})$. Let $L_1\subseteq H$ and choose a nontrivial transvection τ in $DC(\sigma)$ with proper line L_1. Then $\bar{\tau}\in(DC(\sigma))^-=(DC(\sigma))^-\subseteq DC(\bar{\sigma})$. So $\bar{\Sigma}$ and $\bar{\tau}$ commute; Σ and τ cannot anticommute since τ is a transvection and so Σ and τ must commute. Hence $\Sigma L_1=L_1$ for all lines L_1 of H, and so $\Sigma|H=\alpha\cdot 1_H$ for some $\alpha\in\dot{F}$. Now let ϕ be any element of $DC(\sigma)$. Then $\bar{\phi}\in(DC(\sigma))^-\subseteq DC(\bar{\sigma})$. So $\bar{\phi}$ and $\bar{\Sigma}$ commute, and ϕ and Σ cannot anticommute. So ϕ and Σ commute. Thus $\Sigma\in CDC(\sigma)$, and we showed above there is a scalar λ such that the fixed space of $\lambda\cdot\Sigma$ contains H. Now since $\Sigma\in CDC(\sigma)$, we have $\check{\Sigma}\in CDC(\check{\ })$ since the contragredient mapping is an isomorphism of $\mathrm{GL}_m(W)$ onto $\mathrm{GL}_m(W')$ where W' denotes the dual space of W. Dualizing the above reasoning, we see that there is a scalar β such that the residual space of $\beta\Sigma$ is contained in L. Since $m\geqq 3$, $\beta=\lambda$. Hence $\bar{\Sigma}=(\beta\Sigma)^-\in\bar{T}(L,H)$. Q.E.D.

COROLLARY. *With the hypotheses of* 2.1, $CDC(\bar{\sigma})$ *is abelian.*

DEFINITION. An element of $\bar{\sigma}$ of $\mathrm{PGL}(W)$ is called a projective shearing if there is a shearing $\Sigma\in\mathrm{GL}(W)$ such that $\bar{\Sigma}=\bar{\sigma}$.

2.2. *Let $n\geqq 3$ and G_1 be any subgroup of $U_n(V)$ that has enough projective transvections. Let $\bar{\sigma}_1$ and $\bar{\sigma}_2$ be nontrivial projective shearings in G_1. Then $C(\bar{\sigma}_1)=C(\bar{\sigma}_2)$ iff $L_1=L_2$ where L_i is the proper line of $\bar{\sigma}_i$.*

PROOF. Suppose $L_1=L_2$. Then it follows from the proof of 2.9 of [5] that $C(\sigma_1)=C(\sigma_2)$, and so $(C(\sigma_1))^-=(C(\sigma_2))^-$. Since nothing can anticommute with a shearing for $n\geqq 3$ we see $C(\bar{\sigma}_1)=C(\bar{\sigma}_2)$.

Conversely, suppose $C(\bar{\sigma}_1)=C(\bar{\sigma}_2)$. Since nothing can anticommute with a shearing, this implies $C(\sigma_1)=C(\sigma_2)$ where the last two centralizers are taken in the group Δ_1. Suppose $L_1\neq L_2$. Then the hyperplanes L_1^* and L_2^* are distinct so there is an isotropic line L which is in L_1^* but not in L_2^*. Let σ_3 be a nontrivial transvection in Δ_1 with proper line L. Then $\sigma_3\in C(\sigma_1)$ but $\sigma_3\notin C(\sigma_2)$, which contradicts $C(\sigma_1)=C(\sigma_2)$. Hence $L_1=L_2$. Q.E.D.

2.3. *Let $m\geqq 4$ and let Λ be an isomorphism of G onto G_1, $\Lambda:G\to G_1$. Let $\bar{\sigma}$ be a projective transvection in G with proper line L and proper hyperplane H. Then $\Lambda\bar{\sigma}$ is a projective shearing.*

PROOF. Let $\bar{\Sigma}=\Lambda\bar{\sigma}$; we can assume $\bar{\sigma}\neq\bar{1}$ so $\bar{\Sigma}\neq\bar{1}$. Hence there is an isotropic line $L_1=F_1a$ such that $\Sigma L_1\neq L_1$. Let $T=\tau_{a,\lambda}$ be a nontrivial transvection in Δ_1 with proper line L_1. By 1.1 we can assume Σ and $T\Sigma^{-1}T^{-1}$ do not commute; a dimension argument shows they can not anticommute. Put $h=\Sigma T\Sigma^{-1}T^{-1}$, put $\Lambda\bar{\tau}=\bar{T}$ and set $f=\sigma\tau\sigma^{-1}\tau^{-1}$. Clearly $\Lambda\bar{f}=\bar{h}$; and since $\bar{\Sigma}$ and $\bar{T}\bar{\Sigma}^{-1}\bar{T}^{-1}$ do not commute, $\bar{\sigma}$ and $\bar{\tau}\bar{\sigma}^{-1}\bar{\tau}^{-1}$ do not commute. Thus σ and $\tau\sigma^{-1}\tau^{-1}$ do not commute.

Now $h=(\Sigma T\Sigma^{-1})T^{-1}$ is the product of two transvections with distinct proper lines, ΣL_1 and L_1 respectively. Hence h has residual space the plane $R=L_1+\Sigma L_1$. So R is either a totally degenerate plane or regular plane. Now $\Sigma T\Sigma^{-1}=\tau_{\Sigma a,\lambda}$ and $T^{-1}=\tau_{a,-\lambda}$. From these formulae it follows $\Sigma T\Sigma^{-1}$ and T^{-1} both act on $R=F_1a+\Sigma F_1a$. Thus either $\Sigma T\Sigma^{-1}$ and T^{-1} both induce 1_R, or they induce nontrivial transvections on R with distinct proper lines. (Depending on whether $(a,\Sigma a)=0$ or $(a,\Sigma a)\neq 0$.) In any case $h|R\neq -1_R$ since $\chi(F)\neq 2$. By 1.7 of [5], $h^2\neq 1_V$ and surely $h^2\neq\gamma\cdot 1_V$, with $\gamma\neq 1$. So $\bar{h}^2\neq\bar{1}_V$; thus $\bar{f}^2\neq\bar{1}_V$ and so $f^2\neq 1_V$.

Now let us compute the fixed and residual spaces R_2 and P_2 of f. We know $R_2\subseteq L+\tau L$ and $P_2\supseteq H\cap\tau H$ by 1.1 of [5]. We will show $\tau H\neq H$, $\tau L\neq L$, $R_2=L+\tau L$, $P_2=H\cap\tau H$, $R_2\nsubseteq P_2$. We have shown above σ and $\tau\sigma^{-1}\tau^{-1}$ do not commute. Their spaces are $L\subseteq H$ and $\tau L\subseteq\tau H$ respectively. Hence, $L\nsubseteq\tau H$ or $\tau L\nsubseteq H$ by 1.6 of [5]. So in particular $\tau L\neq L$ and $\tau H\neq H$. So $V=\tau H+H$ and so $R_2=L+\tau L$ by 1.2 of [5]. It follows that $P_2=\tau H\cap H$. Finally $R_2=L+\tau L\nsubseteq H\cap\tau H=P_2$. We have now shown f satisfies the hypotheses of 1.1 of [8]. Hence by 1.1 of [8], $CDC(\bar{f})$ contains every element $\bar{\sigma}$ of G such that the residual space of σ is contained in R_2 and such that the fixed space of σ contains P_2. From this it follows both $\bar{\sigma}$ and $\bar{\tau}\bar{\sigma}^{-1}\bar{\tau}^{-1}$ are in $CDC(\bar{f})$, and since these latter two elements do not commute, it follows $CDC(\bar{k})$ is nonabelian. Since $CDC(\bar{f})$ is nonabelian, $CDC(\Lambda\bar{f})=CDC(\bar{f})$ is nonabelian and so by 1.5, R is a regular plane, and therefore a hyperbolic plane.

Thus $\Lambda\bar{\sigma}\in CDC(\Lambda\bar{f})=CDC(\bar{h})$ since $\bar{\sigma}\in CDC(\bar{f})$. So by 1.3, $\bar{\Sigma}=\Lambda\bar{\sigma}\in CDC(\bar{h})\subseteq(E(R))^-$ where R is the residual space of h. Hence we may assume Σ has residual space a line or has residual space the hyperbolic plane R. If R is the residual space of Σ and $\Sigma|R$ is a scalar, then since $CDC(\bar{h})\subseteq(E(R))^-$ we see $\bar{\Sigma}$ centralizes $CDC(\bar{h})$ which contradicts the fact $\bar{\sigma}\notin CCDC(\bar{f})$; if R is the residual space of Σ and $\Sigma|R$ is not a scalar, then 1.2 shows $E(R)\subseteq CDC(\Sigma)$ contradicting the corollary of 2.1 that $CDC(\bar{\sigma})$ is abelian. So Σ has residual space a line. Q.E.D.

Application to congruence groups. We now apply Theorem 2.3 to demonstrate the nonisomorphism, under appropriate hypotheses, of the linear, symplectic, and unitary congruence groups. The definition of these congruence groups will be taken as in [10].

2.4. *Let S be a linear congruence group whose associated vector space W has dimension at least* 4. *Let S_1 be a symplectic or a unitary congruence group whose associated form has Witt index at least* 3 *and whose associated field is of characteristic not* 2. *Then S and S_1 are not isomorphic.*

PROOF. Let us define $G=\bar{S}$ and define $G_1=\bar{S}_1$. Now let us suppose that $\Lambda: G \rightarrow G_1$ is an isomorphism. Fix a line L in W and take two nontrivial projective transvections $\bar{\tau}$ and $\bar{\sigma}$ in G with proper spaces $L \subset H$ and $L \subset H_1$ respectively, where H and H_1 are two distinct hyperplanes of W. Then $C(\bar{\tau}) \neq C(\bar{\sigma})$ by 2.9 of [5]. However the composition $\bar{\tau}\bar{\sigma}$ is a nontrivial projective transvection in G, so $\Lambda\bar{\tau}\bar{\sigma}=\Lambda\bar{\tau}\Lambda\bar{\sigma}$ is a transvection or a quasi-symmetry. However $C(\Lambda\bar{\tau}) \neq C(\Lambda\bar{\sigma})$ since $C(\bar{\tau}) \neq C(\bar{\sigma})$. But by the previous theorem $\Lambda\bar{\tau}$ and $\Lambda\bar{\sigma}$ are projective shearings; let them have residual lines L_1 and L_2 respectively. Then $L_1 \neq L_2$ for if $L_1=L_2$, 2.2 would imply $C(\Lambda\bar{\tau})=C(\Lambda\bar{\sigma})$. Since $L_1 \neq L_2$ the composition $\Lambda\bar{\tau}\Lambda\bar{\sigma}$ is neither a projective transvection nor a projective quasi-symmetry, which is a contradiction.

From the fact the groups G and G_1 are not isomorphic, it follows the groups S and S_1 cannot be isomorphic. For any isomorphism between S and S_1 would induce in a natural way an isomorphism between G and G_1. Q.E.D.

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