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ON THE FIRST BOUNDARY VALUE PROBLEM FOR $[h(x, x', t)]' = f(x, x', t)$

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ABSTRACT. The boundary value problem for $[h(x, x', t)]' = f(x, x', t)$ is studied with $x(0)=x(1)=0$. It is assumed that substitution of functions u and v in $L_2(0, 1)$ into h and f produces the functions $h[u(\cdot), v(\cdot), \cdot]$ and $f[u(\cdot), v(\cdot), \cdot]$ in $L_2(0, 1)$ such that this map from $L_2(0, 1) \times L_2(0, 1)$ into $L_2(0, 1) \times L_2(0, 1)$ is hemicontinuous. Existence and uniqueness are shown in $H_0^1(0, 1)$ under the assumption that constants λ and η exist such that

$$[(V - v)[h(U, V, t) - h(u, v, t)] + (U - u)[f(U, V, t) - f(u, v, t)]] \geqq \lambda(V - v)^2 - \eta(U - u)^2$$

whenever t lies between zero and one while u, v, U and V are arbitrary. Also, it is assumed that λ and $\lambda\pi^2 - \eta$ are positive.

1. **Introduction.** Let $I=(0, 1)$, $S_1=R\times[0, 1]$, and $S=R^2\times[0, 1]$ where R represents the real numbers. If $\{h, f\}\subset C(S)$, then the nonlinear boundary value problem

$$[h(x, x', t)]' = f(x, x', t), \qquad t \in I, x(0) = x(1) = 0, \tag{1.1}$$

is a generalization of Euler's equation

$$(d/dt)(\partial L/\partial x') - \partial L/\partial x = 0 \tag{1.2}$$

with fixed endpoints in the calculus of variations. Similarly, Lagrange's equation takes the form

$$(d/dt)(\partial L/\partial x') - \partial L/\partial x = Q \tag{1.3}$$

in nonconservative particle mechanics [4]. Equation (1.1) is often studied in a simplified form. For example, Lees [9] takes $f \in C(S_1)$ and discusses the nonlinear boundary value problem

$$x'' = f(x, t), \qquad t \in I, x(0) = x(1) = 0, \tag{1.4}$$

under the assumed existence of a constant η such that

$$f_1(x, t) \geqq -\eta > -\pi^2 \tag{1.5}$$

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where f_1 represents $\partial f/\partial x$. Similarly, Tippett [**12**] takes $f \in C(S)$ and extends Lees' existence and uniqueness results to the problem

$$x'' = f(x, x', t), \qquad t \in I, x(0) = x(1) = 0, \tag{1.6}$$

under the assumptions that (1.5) is valid and

$$\pi^2 - \eta > \pi \|f_2\|_S \tag{1.7}$$

where f_2 is $\partial f/\partial x'$ and $\|\cdot\|_S$ represents the uniform norm on S. Finally, Komkov [**7**] discusses a similar problem in a physical context. Also, Komkov has applied the results presented below in a recent paper [**8**].

The present paper develops existence and uniqueness of a solution in $H_0^1(I)$ for the nonlinear boundary value problem in (1.1) based primarily on the assumption that constants η and λ exist such that, whenever $\{(u, v, t), (U, V, t)\} \subset S$,

$$\begin{aligned}(V - v)[h(U, V, t) - h(u, v, t)] + (U - u)[f(U, V, t) - f(u, v, t)]\\ \geqq \lambda(V - v)^2 - \eta(U - u)^2,\end{aligned} \tag{1.8}$$

where

$$\lambda\pi^2 - \eta > 0, \tag{1.9}$$

and

$$\lambda > 0. \tag{1.10}$$

It is interesting to note that (1.8) reduces to (1.5) if we observe that $h(u, v, t) = v$, $f(u, v, t)$ becomes $f(u, t)$, λ can be taken to be unity, and the above inequalities can be written as

$$(f(U, t) - f(u, t))/(U - u) \geqq -\eta > -\pi^2. \tag{1.11}$$

Thus, (1.1) and (1.8) represent a rather natural generalization of Lees' [**9**] discussion.

It should be noted that an interesting alternate for (1.8) is available if $\{h_1, h_2, f_1, f_2\} \subset C(S)$. In this case, fix $\{(u, v, t), (U, V, t)\} \subset S$ and let

$$H(\xi, t) = h[\xi U + (1 - \xi)u, \xi V + (1 - \xi)v, t]. \tag{1.12}$$

Then $\partial H/\partial \xi$ can be written as

$$H_1(\xi, t) = (U - u)h_1 + (V - v)h_2, \tag{1.13}$$

and

$$h(U, V, t) - h(u, v, t) = \int_0^1 H_1(\xi, t)\, d\xi. \tag{1.14}$$

Similarly,

$$f(U, V, t) - f(u, v, t) = \int_0^1 F_1(\xi, t)\, d\xi, \tag{1.15}$$

and (1.8) is implied by the form

$$\int_0^1 (U - u, V - v)\begin{pmatrix} f_1 + \eta & f_2 \\ h_1 & h_2 - \lambda \end{pmatrix}\begin{pmatrix} U - u \\ V - v \end{pmatrix} d\xi \geqq 0. \tag{1.16}$$

It is clear that a sufficient condition for the validity of (1.16) is given by the assumption that

$$(\alpha, \beta)\begin{pmatrix} f_1(u, v, t) + \eta & f_2(u, v, t) \\ h_1(u, v, t) & h_2(u, v, t) - \lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \geqq 0 \tag{1.17}$$

for every $(u, v, t) \in S$ and $(\alpha, \beta) \in R^2$.

Although certain continuity assumptions on f and h will be needed in the existence proof (§4) for (1.1), it should be noted that (1.8) through (1.10) are the essential assumptions. The inequalities in (1.16) and (1.17) are given above as easily checked sufficient conditions that (1.8) is applicable in a particular example. Conditions (1.8) through (1.10) may well be more natural than the Lipschitz conditions that are used in the usual approach **[1]** to existence and uniqueness for (1.6). Indeed, the work of Browder **[2]** seems to suggest the present approach.

Since each element of $H_0^1(0, 1)$ is absolutely continuous, the solution of (1.1) in $H_0^1(0, 1)$ is a classical solution in the sense that it has as many derivatives (namely, one derivative) as (1.1) explicitly requires.

Although it seems possible that Galerkin methods **[3]** might be used to develop approximate solutions of (1.1), no such work has been done on this in so far as we know.

2. **Preliminaries.** Let $C_0^\infty(I)$ represent the infinitely differentiable functions with support in the interval I. Let $(\cdot, \cdot)$ denote the $L_2(I)$ inner product and let

$$(x, y)_H = (x, y) + (x', y') \tag{2.1}$$

be the inner product leading to the norm defined by

$$\|x\|_H^2 = \|x\|^2 + \|x'\|^2. \tag{2.2}$$

Then, $H_0^1(I)$ is the well-known Hilbert space obtained by completion of $C_0^\infty(I)$ with respect to the latter norm. Also, recall **[5]** that

$$\pi\|x\| \leqq \|x'\|, \qquad x \in C'[0, 1], x(0) = x(1) = 0. \tag{2.3}$$

If x is a real-valued function with x' defined on [0, 1], write $f(x; t)$ and $h(x; t)$ for $f[x(t), x'(t), t]$ and $h[x(t), x'(t), t]$, respectively. Also, let $h'(x; t)$ represent the derivative of $h[x(t), x'(t), t]$ with respect to t. Introduce the notation

$$c(t) = (\cosh t)/(\sinh 1)^{1/2} \tag{2.4}$$

and

$$s(t) = (\sinh t)/(\sinh 1)^{1/2}. \tag{2.5}$$

If $y \in C_0^\infty(I)$, formal integration by parts leads to

$$(y, f(x; \cdot)) - (y, h'(x; \cdot)) = (y, f(x; \cdot)) + (y', h(x; \cdot)). \tag{2.6}$$

LEMMA 2.1. *Let $\{u, v\} \subset L_2(I)$ and define the linear functional $\alpha(u, v; \cdot)$ on $H_0^1(I)$ by*

$$\alpha(u, v; y) = (y, u) + (y', v). \tag{2.7}$$

Then, $\alpha(u, v; \cdot)$ is in the space $[H_0^1(I)]^*$ of continuous linear functionals on $H_0^1(I)$.

PROOF. It suffices to show that $\alpha(u, v; \cdot)$ is bounded. Use the Cauchy-Schwarz inequality in $L_2(I)$ and in R^2 to obtain, for $y \in H_0^1(I)$,

$$\begin{aligned} |\alpha(u, v; y)| &\leqq |(y, u)| + |(y', v)| \\ &\leqq \|y\| \cdot \|u\| + \|y'\| \cdot \|v\| \leqq \|y\|_H (\|u\|^2 + \|v\|^2)^{1/2}, \end{aligned} \tag{2.8}$$

and the proof is complete.

LEMMA 2.2. *Let $\{u, v\} \subset L_2(I)$ and define $\alpha(u, v; \cdot)$ by* (2.7). *Define the function*

$$z(t) = -s(t-1)I_1(t) - s(t)I_2(t) \tag{2.9}$$

where

$$I_1(t) = \int_0^t [s(\tau)u(\tau) + c(\tau)v(\tau)]\, d\tau \tag{2.10}$$

and

$$I_2(t) = \int_t^1 [s(\tau-1)u(\tau) + c(\tau-1)v(\tau)]\, d\tau. \tag{2.11}$$

Then, $z' \in L_2(I)$,

$$(y, z)_H = \alpha(u, v; y), \qquad y \in H_0^1(I), \tag{2.12}$$

and it follows that $z \in H_0^1(I)$.

MOTIVATION. Formal manipulation of (2.7) and (2.12) leads to

$$(2.13)\qquad \begin{aligned}(y,u)+(y',v)&=(y,u)-(y,v')=(y,u-v')\\ &=(y,z)+(y',z')=(y,z)-(y,z'')=(y,z-z''),\end{aligned}$$

and (2.9) is the solution (by Green's functions) of the boundary value problem

$$(2.14)\qquad z''-z=v'-u,\qquad z(0)=z(1)=0.$$

PROOF. Use (2.9) to calculate z' in the form

$$(2.15)\qquad \begin{aligned}z'(t)&=-c(t-1)I_1(t)-c(t)I_2(t)-s(t-1)[s(t)u(t)+c(t)v(t)]\\ &\quad+s(t)[s(t-1)u(t)+c(t-1)v(t)]\\ &=v(t)-c(t-1)I_1(t)-c(t)I_2(t)\end{aligned}$$

to see that $z'\in L_2(I)$. If $y\in H_0^1(I)$, observe that

$$(2.16)\qquad (yc)'=yc'+y'c=ys+y'c$$

and calculate

$$(2.17)\qquad \begin{aligned}(y,z)_H-(y',v)&=(y,z)+(y',z'-v)\\ &=-\int_0^1\{[y(\theta)c(\theta-1)]'I_1(\theta)+[y(\theta)c(\theta)]'I_2(\theta)\}\,d\theta\\ &=\int_0^1\{c(\theta-1)I_1'(\theta)+c(\theta)I_2'(\theta)\}y(\theta)\,d\theta\\ &=\int_0^1\{[c(\theta-1)s(\theta)-c(\theta)s(\theta-1)]u(\theta)\\ &\qquad+[c(\theta-1)c(\theta)-c(\theta)c(\theta-1)]v(\theta)\}y(\theta)\,d\theta\\ &=\int_0^1 u(\theta)y(\theta)\,d\theta=(y,u).\end{aligned}$$

Since (2.17) implies (2.12), the Riesz representation theorem implies $z\in H_0^1(I)$, and the proof is complete.

DEFINITION 2.1. Suppose h and f in (1.1) are such that $x'\in L_2(I)$ implies that $\{h(x;\cdot),f(x;\cdot)\}\subset L_2(I)$. Define $z(x;\cdot)\in H_0^1(I)$ when $x\in H_0^1(I)$ by substitution of $f(x;\cdot)$, $h(x;\cdot)$ and $z(x;\cdot)$ in (2.7) through (2.12) for u, v, and z, respectively. Let $Z(x)=z(x;\cdot)$ define a map Z in $H_0^1(I)$. Then, x will be said to be a solution of (1.1) if and only if $Z(x)$ is zero.

3. **Uniqueness.** The uniqueness of $x\in H_0^1(I)$ satisfying (1.1) is based on the strong monotonicity of the mapping Z.

THEOREM 3.1. *Suppose that Definition* 2.1 *is applicable and that* (1.8) *through* (1.10) *are valid. Then Z is strongly monotone in the sense that* $\{x, y\} \subset H_0^1(I)$ *implies that*

$$(x - y, Z(x) - Z(y))_H \geqq K\|x - y\|_H^2 \tag{3.1}$$

where the positive constant K is given by ($a \wedge b$ *means the smaller of a and b*)

$$K = (1 + \pi^{-2})^{-1/2}[\lambda \wedge (\lambda - \eta\pi^{-2})]. \tag{3.2}$$

Thus, (1.1) *has at most one solution in* $H_0^1(I)$.

PROOF. It is convenient to let Δx, ΔZ, Δf, and Δh denote $x-y$, $Z(x)-Z(y)$, $f(x;\cdot)-f(y;\cdot)$, and $h(x;\cdot)-h(y;\cdot)$, respectively. Since integration is a linear operation, Definition 2.1 and (2.9) through (2.11) lead to

$$(\Delta Z)(t) = -s(t-1)I_1(\Delta f, \Delta h; t) - s(t)I_2(\Delta f, \Delta h; t) \tag{3.3}$$

where $I_j(\Delta f, \Delta h; t)$ denotes the result obtained in setting $u=\Delta f$ and $v=\Delta h$ in $I_j(t)$. Lemma 2.2 implies that

$$(\Delta x, \Delta Z)_H = \alpha(\Delta f, \Delta h; \Delta x) = (\Delta x, \Delta f) + (\Delta x', \Delta h). \tag{3.4}$$

Now, apply (1.8) to obtain

$$(\Delta x, \Delta Z)_H \geqq \lambda \|\Delta x'\|^2 - \eta \|\Delta x\|^2. \tag{3.5}$$

If $\eta>0$, use (2.3) to write

$$(\Delta x, \Delta Z)_H \geqq (\lambda - \eta\pi^{-2}) \|\Delta x'\|^2. \tag{3.6}$$

If $\eta\leqq 0$, ignore η to write (3.5) in the form

$$(\Delta x, \Delta Z)_H \geqq \lambda \|\Delta x'\|^2. \tag{3.7}$$

Since (2.3) implies that

$$\|u\|_H \leqq (1 + \pi^{-2})^{1/2} \|u'\|, \qquad u \in H_0^1(I), \tag{3.8}$$

(3.1) is valid with K defined by (3.2). Finally, if x and y are solutions of (1.1) in $H_0^1(I)$, then

$$(u, \Delta Z) = 0, \qquad u \in H_0^1(I). \tag{3.9}$$

Choose $u=\Delta x$, observe that $K>0$ as a consequence of (1.9) and (1.10), and the proof is complete.

4. **Existence.** The existence of a solution $x \in H_0^1(I)$ for (1.1) will be based on a result given by Browder [**2**].

DEFINITION 4.1. Let T map a Banach space X into a Banach space Y. T is said to be hemicontinuous (=feebly continuous) if and only if each restriction of T to a closed segment in X is a continuous map into Y with respect to the weak topology in Y.

THEOREM 4.1. *Let T be a hemicontinuous, monotone* (K *in* (3.1) *may be zero*) *map of a reflexive Banach space X into its dual space X^* such that*

$$(Tu, u)/\|u\|_X \to \infty \quad as \ \|u\|_X \to \infty. \tag{4.1}$$

Then, T maps X onto X^*.

PROOF. [**2**, pp. 18–24].

ASSUMPTION 4.1. The calculation of $f(y; \cdot)$ and $h(y; \cdot)$ for $y \in H_0^1(I)$ represents two hemicontinuous maps of $H_0^1(I)$ into $L_2(I)$.

THEOREM 4.2. *Let f and h be such that Assumption* 4.1 *applies and* (1.8) *through* (1.10) *are valid. Then the mapping Z in Definition* 2.1 *maps $H_0^1(I)$ onto $H_0^1(I)$.*

PROOF. Theorem 3.1 shows that Z is strongly monotone. In particular, set $y=0$ in (3.1) to write, for $x \in H_0^1(I)$,

$$(x, Z(x))_H \geqq (x, Z(0))_H + K\ \|x\|_H^2. \tag{4.2}$$

Since $K>0$, and

$$|(x, Z(0))_H|/\|x\|_H \leqq \|Z(0)\|_H; \tag{4.3}$$

$$(x, z(x))_H/\|x\|_H \to \infty \quad \text{as } \|x\|_H \to \infty. \tag{4.4}$$

Observe that a comparison of (2.9) and (2.15) shows that Z has the same type of continuity as the mapping suggested by $x \to h(x; \cdot)$. Since $[H_0^1(I)]^* = H_0^1(I)$, Theorem 4.1 implies that Z is surjective, and the proof is complete.

THEOREM 4.3. *Let Z satisfy the hypotheses above. Then,* (1.1) *has a unique solution in $H_0^1(I)$.*

PROOF. Use Theorem 4.2 to obtain $x \in H_0^1(I)$ such that $Z(x)=0$, Theorem 3.1 shows uniqueness, and the proof is complete.

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