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ON THE CONSISTENCY THEOREM IN MATRIX SUMMABILITY

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ABSTRACT. We give a generalization of the consistency theorem for bounded convergence fields. The space c of convergent sequences is replaced by a space of more general type. Applications of the generalized consistency theorem are made to multiplicative summability theory. In particular we give conditions under which a generalized convergence field is an algebra.

1. **Introduction.** In 1933, Mazur and Orlicz **[M-O$_1$]** announced their famous result that if the bounded convergence field of a regular matrix A is contained in that of another regular matrix B, then the matrices are consistent on the first field, i.e., A-lim $x=B$-lim x whenever x is in the bounded convergence field of A. (Their proof, based on functional analysis, was published in **[M-O$_2$]**.) Later Brudno **[B]** gave a complicated but elementary proof. Independently of each other, G. M. Petersen **[P$_1$]** and Erdös-Piranian **[E-P]** were able to isolate a basic principle used implicitly by Brudno, and to use this principle to give easy proofs of the consistency theorem.

In §2 we give a formulation of the basic principle which, combined with certain facts concerning the topology of the space $\beta N \backslash N$ (to be explained in §2), yields in §3 a generalization of the consistency theorem. In this generalization the space c of convergent sequences is replaced by a space of more general type, subject to certain conditions. Most importantly the space must, in a sense to be specified, be 'small' as a subspace of the space of bounded sequences. As an application of the generalized consistency theorem we derive in §4 generalizations of known results on the multiplicative behavior of regular matrices.

Basic to our method is the device of representing a regular matrix as a linear operator on $C(\beta N \backslash N)$, as explained below. (See **[A-B]**, **[A]**.)

2. **The basic lemma.** Our version of the basic principle mentioned in the introduction is given in Lemma 2.4. We have stated it in a form

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convenient for our own use, but the details of the proof are gleaned from [$\mathbf{P}_1$], [$\mathbf{P}_2$] and [E-P]. For completeness we give a detailed proof.

2.1. NOTATION. $C^*(N)$ is the space of bounded real valued functions on the positive integers N. If $f \in C^*(N)$, f' is its extension to βN, the Stone-Čech compactification of N, and f^* the restriction of f' to the compact space $\beta N \backslash N$. If $V \subset N$, V' is its closure in βN, and $V^* = V' \cap (\beta N \backslash N)$. $V^* \neq \varnothing$ iff V is infinite, and sets of the form V^* ($V \subset N$) are a basis of clopen (=open and closed) sets for the topology of $N^* = \beta N \backslash N$. (See [R] for details.)

If $C(N^*)$ is the space of continuous real valued functions on N^*, then it is isomorphic to the quotient space $C^*(N)/c_0$, where c_0 is the space of real functions on N with limit 0. If $A = (a_{mn})$ is a regular matrix operator on $C^*(N)$, then $A(c_0) \subset c_0$, and hence A induces an operator A^* on $C(N^*)$ by the formula $A^*(f^*) = (Af)^*$ ($f \in C^*(N)$). Note that $f^* = 0$ iff $f \in c_0$, in which case $Af \in c_0$ as well. Hence $f^* = 0$ implies $A^*(f^*) = (Af)^* = 0$, and so A^* is a well-defined linear operator on $C(N^*)$. From regularity of A it follows easily that $(A^*e)(x) = 1$ for all $x \in N^*$, where $e(x) = 1$ for all $x \in N^*$. If C_A is the bounded convergence field of A, let $(C_A)^* = \{f^* : f \in C_A\}$. Then $(C_A)^* = \{f \in C(N^*) : A^*f = \text{constant}\}$.

2.2. DEFINITION. M_A is the set of $g \in C^*(N)$ such that $\lim$ $(n \to \infty)$ $\{A(gf)(n) - (Ag)(n)(Af)(n)\} = 0$ for all $f \in C^*(N)$. $(M_A)^* = \{g^* : g \in M_A\}$.

In [A] it was shown that M_A and $(M_A)^*$ are Banach algebras, and that $g \in (M_A)^*$ iff $A^*(gf) = (A^*g)(A^*f)$ for all $f \in C(N^*)$.

2.3. DEFINITION. A linear subspace L of $C(N^*)$ is called *large* if for each nonvoid clopen $V \subset N^*$, there exists $f \in L$ such that $f(x) = 0$ and $f(y) = 1$ for some $x \in V$ and $y \in V$.

2.4. LEMMA. *Let A^r ($r \in N$) be regular matrices, M^r the corresponding algebras as in 2.2, and $M = \bigcap \{M^r : r \in N\}$. Then M^* is large, and so is $(A^r)^*(M^*)$ for each r (where $(A^r)^*(M^*) = \{(A^r)^*(f) : f \in M^*\}$).*

PROOF. First we consider the case of a single matrix $A = (a_{mn})$, which we assume without loss of generality to be in *truncated form*, i.e., there exist $m(1) < m(2) < \cdots$ and $n(1) < n(2) < \cdots$ such that $m \in [m(k), m(k+1))$ implies $a_{mn} = 0$ whenever $n \notin [n(k), n(k+2))$. (See [$\mathbf{P}_2$, p. 82] or [E-P].) We may also assume each row sum is 1. Let $t(n)$ be any bounded sequence of reals such that $\varepsilon(n) = t(n+1) - t(n) \to 0$, and define $g \in C^*(N)$ by $g(r) = t(k)$ whenever $r \in [n(k), n(k+1))$. We show that $g \in M_A$. If $f \in C^*(N)$ and $m \in [m(k), m(k+1))$,

$$(*) \quad \begin{aligned} \sum_j a_{mj} g(j) f(j) &= t(k) \sum_{j=n(k)}^{n(k+1)-1} a_{mj} f(j) + t(k+1) \sum_{j=n(k+1)}^{n(k+2)-1} a_{mj} f(j) \\ &= t(k) \sum_{j=n(k)}^{n(k+2)-1} a_{mj} f(j) + \varepsilon(k) \sum_{j=n(k+1)}^{n(k+2)-1} a_{mj} f(j). \end{aligned}$$

Letting $f=1$ and recalling that row sums are 1, we have

$$t(k)=\sum_j a_{mj}g(j)-\varepsilon(k)\sum_{j=n(k+1)}^{n(k+2)-1} a_{mj}.$$

Putting this value of $t(k)$ into (*) gives

$$\sum_j a_{mj}g(j)f(j)=\left[\sum_j a_{mj}g(j)-\varepsilon(k)\sum_{j=n(k+1)}^{n(k+2)-1} a_{mj}\right]\cdot\left[\sum_{j=n(k)}^{n(k+2)-1} a_{mj}f(j)\right]+\varepsilon(k)\sum_{j=n(k+1)}^{n(k+2)-1} a_{mj}f(j).$$

It follows that

$$\sum_j a_{mj}g(j)f(j)-\Big(\sum_j a_{mj}g(j)\Big)\Big(\sum_j a_{mj}f(j)\Big)\to 0.$$

Hence $g\in M_A$.

Let $V\subset N$ be infinite. We shall find $g\in M_A$ such that g^* takes the values 0 and 1 on the clopen set V^*. But V must meet an infinite number of the intervals $[n(k), n(k+1))$, so by choosing $t(k)$ to be 0 and 1 infinitely often for k such that V meets $[n(k), n(k+1))$, and also so that $t(n+1)-t(n)\to 0$, we obtain (as above) a g as desired (cf. [$\mathbf{P}_1$]). Clearly, the extension g^* takes the values 0 and 1 on V^*. A similar construction will give g such that the transform Ag does the same.

The above construction of the function g depended only on the particular way the matrix A was truncated. Hence to complete the proof it suffices to show that the matrices A^r ($r\in N$) can be 'simultaneously truncated' (cf. [$\mathbf{P}_2$, p. 95]). We give the first few steps of an algorithm for achieving this. Let $m(1)=n(1)=1$, and let $n(2)>n(1)$. By regularity of A^1 and A^2 we choose $m(2)>m(1)$ such that $m\geqq m(2)$ implies $\sum\{|a^r_{mn}|:n\leqq n(2)\}<\frac{1}{2}$ $(r=1,2)$. Now choose $n(3)>n(2)$ such that $m(1)\leqq m<m(2)$ implies $\sum\{|a^r_{mn}|:n(3)\leqq n\}<\frac{1}{2}$ $(r\in 1,2)$. Choose $m(3)>m(2)$ such that $m\geqq m(3)$ implies $\sum\{|a^r_{mn}|:n\leqq n(3)\}<\frac{1}{3}$ $(r=1,2,3)$. Choose $n(4)>n(3)$ such that $m(2)\leqq m<m(3)$ implies $\sum\{|a^r_{mn}|:n(4)\leqq n\}<\frac{1}{3}$ $(r=1,2,3)$. And so on. (Cf. the picture in [**E-P**].) Every matrix A^r will have truncated form beginning with some row, and it is easy to see that rows may be adjusted to have sum 1 without affecting limiting behavior at infinity.

3. **The consistency theorem.** If A is an infinite matrix such that $A(c_0)\subset c_0$, then it is easy to see that regularity of A is equivalent with $A^*e=e$, where e is the unit function in $C(N^*)$. We have also noted that if C_A is the bounded convergence field of A, then $(C_A)^*=\{f\in C(N^*):A^*f=\text{const}\}$, and $C_A\subset C_B$ iff $(C_A)^*\subset(C_B)^*$. Thus facts about matrix operators

on $C^*(N)$ translate into facts about the induced operators on $C(N^*)$ and vice versa. For the remainder of the paper our results are stated in terms of the induced operators on $C(N^*)$, leaving the relevant translations to the reader.

3.1. TOPOLOGICAL FACTS. (a) A nonvoid G_δ set in N^* always has nonvoid interior. (b) A nonvoid open set in N^* cannot be expressed as a union of a collection of cardinality $\aleph_1$ of closed nowhere dense sets.

For fact (a), see **[R]**. (b) follows from (a) **[Pl**, p. 46]; and, as Plank notes, this fact is implicit in **[R]**.

3.2. DEFINITION. Let L be a linear subspace of $C(N^*)$. Two points x, $y \in N^*$ are *equivalent under* L if $f(x)=f(y)$ for all $f \in L$. L is called *small* if there exist at most $\aleph_1$ equivalence classes under L.

Since N^* has 2^c points **[R]**, $C(N^*)$ is not small. Obviously, the space of constant functions is small. A less trivial example: let $f \in C(N^*)$ have finite or countable range, and let L be the Banach subalgebra of $C(N^*)$ generated by f.

If we assume the continuum hypothesis $c=\aleph_1$, then the Banach subalgebra of $C(N^*)$ generated by any finite or countable collection of elements is small.

3.3. DEFINITION. If L is a linear subspace of $C(N^*)$ and A a regular matrix, $C_{AL}=\{f \in C(N^*): A^*f \in L\}$. Note that if L=constant functions, then $C_{AL}=(C_A)^*$, where C_A is the bounded convergence field of A and $(C_A)^*=\{f^*: f \in C_A\}$.

3.4. CONSISTENCY THEOREM. *Let A and B be regular matrices and L a small linear subspace of $C(N^*)$. Suppose that for each $k \in L$, there exists $h \in L$ such that $A^*h=B^*h=k$. Then $C_{AL} \subset C_{BL}$ implies $A^*f=B^*f$ for all $f \in C_{AL}$.*

PROOF. Let $f_1 \in C_{AL}$, $A^*f_1=k$, and suppose $B^*f_1=p \neq k$. We show that this implies $C_{AL} \not\subset C_{BL}$. Choose $h \in L$ such that $A^*h=B^*h=k$, and let $f=f_1-h$. Then $A^*f=0 \in L$, and $B^*f=p-k=j \in L\backslash\{0\}$. Choose real $s \neq 0$ such that $j^{-1}(s)$ is nonvoid. Then $j^{-1}(s)$ is a closed G_δ set, so 3.1 (a) implies $j^{-1}(s)$ contains a nonvoid clopen set W. Now W meets at most $\aleph_1$ equivalence classes under L, so by 3.1 (b) at least one such class contains a clopen set V. By Lemma 2.4 there exists $g \in (M_A)^* \cap (M_B)^*$ such that B^*g takes the values 0 and 1 on V. Then $A^*(fg)=(A^*f)(A^*g)=0 \in L$, while $B^*(fg)=(B^*f)(B^*g)=s(B^*g)$ on V. Since B^*g is not constant on V and V is contained in a set of constancy for L, it follows that $B^*(fg) \notin L$. Hence $fg \in C_{AL} \backslash C_{BL}$, and $C_{AL} \not\subset C_{BL}$.

3.5. REMARK. A simple situation where the hypotheses of 3.4 hold is when $A^*(L)=B^*(L)=L$, and $A^*f=B^*f=f$ for all $f \in L$. By regularity of A and B, this includes the case where L=constant functions.

4. **Applications to multiplicative summability theory.** For the classical case L=constant functions, Theorem 4.2 was proved by Hill and Sledd [**H-S**, p. 412]. The main idea in the proof of 4.1 occurs in their paper as well.

4.1. THEOREM. *Let L be a small subalgebra of $(M_A)^*$ such that* (a) $e \in L$, (b) *if $f \in L$ and f has no zeros, then $f^{-1} \in L$, and* (c) $L \subset (A^*)(L)$. *If $gC_{AL} \subset C_{AL}$, then $A^*(fg)=(A^*f)(A^*g)$ for all $f \in C_{AL}$.*

PROOF. Since $e \in C_{AL}$, we have $g \in C_{AL}$, so $A^*g \in L$. Assume

Case I. $A^*g=k$ has no zeros. Then $k^{-1} \in L$, and by (c) there is $h \in L$ with $A^*h=k^{-1}$. Let the operator B^* be defined on $C(N^*)$ by $B^*f=A^*(hgf)=(A^*h)A^*(gf)=k^{-1}A^*(gf)$, the next to the last equality because $h \in L \subset (M_A)^*$. The operator B^* on $C(N^*)$ is induced by the matrix operator B on $C^*(N)$ defined as follows: let g_1 and h_1 in $C^*(N)$ be such that $g=(g_1)^*$ and $h=(h_1)^*$. Then $Bf=A(h_1g_1f)$ $(f \in C^*(N))$. B is regular, since $B^*e=k^{-1}A^*g=k^{-1}k=e$.

We show that the pair A, B satisfies the conditions of Theorem 3.4. First, $C_{AL} \subset C_{BL}$, because $f \in C_{AL}$ implies $fg \in C_{AL}$ implies $B^*f=A^*hA^*(fg) \in LL \subset L$ (L is an algebra) implies $f \in C_{BL}$. Secondly, if $u \in L$, then by (c) there is $v \in L \subset (M_A)^*$ with $A^*v=u$, and then $B^*v=k^{-1}A^*(gv)=k^{-1}(A^*g)(A^*v)=k^{-1}ku=u$. Thus $A^*v=B^*v=u$.

It now follows from Theorem 3.4 that for all $f \in C_{AL}$, $A^*f=B^*f=k^{-1}A^*(gf)$, or $A^*(gf)=(A^*g)(A^*f)$.

Case II. If $A^*g=k$ and k has zeros, let ρ be a constant such that $A^*(g+\rho e)=k+\rho e$ has no zeros, and let $g'=g+\rho e$. Then $g'C_{AL} \subset C_{AL}$, so by what we have already shown, $A^*(g'f)=(A^*g')(A^*f)$ for all $f \in C_{AL}$. Hence

$$\begin{aligned} A^*(gf) + \rho(A^*f) &= A^*(gf + \rho f) = A^*(g'f) = (A^*g')(A^*f) \\ &= (A^*g + \rho)A^*f = (A^*g)(A^*f) + \rho(A^*f), \end{aligned}$$

or $A^*(gf)=(A^*g)(A^*f)$.

4.2. THEOREM. *Let L be as in* 4.1, *and assume $A^* \geqq 0$. The following are equivalent*:

(a) *$A^*(fg)=(A^*f)(A^*g)$ for all f and g in C_{AL}*;

(b) $C_{AL} \subset (M_A)^*$;

(c) *C_{AL} is an algebra.*

PROOF. (a) *implies* (b). If $f \in C_{AL}$, then (a) implies $A^*(f^2)=(A^*f)^2$. Since $A^* \geqq 0$, [**A**, Theorem 1.3] implies $f \in (M_A)^*$.

(b) *implies* (c). Let f and $g \in C_{AL}$. If (b) holds, then $A^*(fg)=(A^*f)(A^*g) \in LL \subset L$, since L is an algebra. Hence $fg \in C_{AL}$, so C_{AL} is an algebra.

(c) *implies* (a). This follows from 4.1.

4.3. REMARKS. (a) Positivity of A^* was used only for the implication '(a) implies (b)'. (b) In the case L=constant functions, it was shown in [**A**, Theorem 1.5] that $C_{AL}\cap(M_A)^*$ is the set of functions constant on the so-called 'support set' of A^*, and so if $C_{AL}\subset(M_A)^*$, then the convergence field of A has a particularly simple form.

4.4. THEOREM. *Under the hypotheses of* 4.1, *if* $gC_{AL}\subset C_{AL}$ *and* $A^*g=0$, *then* $A^*(g^+)=A^*(g^-)=A^*(|g|)=0$ (*where* $g^+=\max(g,0)$, $g^-=\max(-g,0)$, *and* $|g|=g^++g^-$).

PROOF. By regularity of A, $e\in C_{AL}$, hence $g\in C_{AL}$, $g^2\in C_{AL}$, and by induction $g^k\in C_{AL}$ for all k. Theorem 4.1 implies $A^*(g^k)=0$ for all k. Now as in the proof of the Stone-Weierstrass theorem (N^* is compact), $|g|$ is the uniform limit of polynomials $a_1g+a_2g^2+\cdots+a_kg^k$, so by continuity of A^* we have $A^*(|g|)=0$. But $2g^+=|g|+g$, and $2g^-=|g|-g$, so $0=A^*(g^+)=A^*(g^-)$.

4.5. REMARK. In [**A**, Theorem 1.5] it was shown that if $A^*\geqq 0$, A^* is multiplicative on $(C_A)^*$ iff $(C_A)^*$ consists exactly of elements of $C(N^*)$ which are constant on the so-called support set K of A. Theorem 4.4 leads to an interesting generalization of this. First, some notation: if $p\in N^*$, let m_p be the regular Borel measure representing the functional $f\to(A^*f)(p)$ ($f\in C(N^*)$), K_p the support set of m_p, and K=closure $\bigcup\{K_p:p\in N^*\}$.

4.6. THEOREM. *Let* $A^*\geqq 0$, *and let* L *be a small subalgebra of* $(M_A)^*$ *such that* (a) $e\in L$, (b) *if* $f\in L$ *and* f *has no zeros, then* $f^{-1}\in L$, (c) $A^*f=f$ *for each* $f\in L$. *The following are equivalent*:

(1) C_{AL} *is an algebra*;

(2) C_{AL} *consists exactly of the elements of* $C(N^*)$ *which agree on* K *with some element of* L.

When (1) *and* (2) *hold, then* $A^*f|K=f|K$ *for each* $f\in C_{AL}$.

PROOF. (2) *implies* (1). Since L is an algebra, it is obvious that the set of elements of $C(N^*)$ which agree on K with some element of L is an algebra.

(1) *implies* (2). First, let $f\in C_{AL}$ and $A^*f=k\in L$. Hypothesis (c) implies $A^*(f-k)=0$. Since L is an algebra, 4.4 implies $0=A^*|f-k|$. Hence for each $p\in N^*$, $0=\int|f-k|\,dm_p$. Since m_p is a positive measure, $f=k$ on K_p, and hence on K. On the other hand, if f agrees with $k\in L$ on K, then for each $p\in N^*$, $A^*f(p)=\int f\,dm_p=\int k\,dm_p=A^*k(p)=k(p)$, hence $A^*f=k\in L$, so $f\in C_{AL}$.

The final assertion is now obvious.

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