TANGENT BUNDLES OF HOMOGENEOUS SPACES ARE HOMOGENEOUS SPACES

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Abstract. In this paper we describe how the tangent bundle of a homogeneous space can be viewed as a homogeneous space.

The purpose of this note is to establish a simple result on the structure of the tangent bundle of a homogeneous space. Even though it is both natural and elementary it does not appear to be in the literature.

We shall associate with every Lie group G another Lie group G*, constructed as a semidirect product of G with the Lie algebra of G (the precise definition is given below).

Our result is:

Theorem. If a Lie group G acts transitively and with maximal rank on a differentiable manifold X, then G* acts transitively and with maximal rank on the tangent bundle of X.

Clearly, our result implies that the tangent bundle of a coset space G/H is again a coset space and moreover, is of the form G*/K for some closed subgroup K of G*. We will compute K below.

We now define G* and prove the theorem. Let L be the Lie algebra of G, thought of as the tangent space of G at the identity. For each g ∈ G, we let ad(g) denote the differential at the identity of the inner automorphism x → gxg^−1 of G. Thus ad is a (not necessarily one-to-one) homomorphism of G into the group of linear automorphisms of L. We define G* as the product manifold L × G, with the group operation given by

\[(a, g) \cdot (a', g') = (a + ad(g)(a'), gg').\]

The verification that G* is a group is trivial and will be omitted. Also, it is clear that the operation defined by (1) is differentiable, so that G* is a Lie group.

Now, let G act differentiably on a manifold X. For each x ∈ X, let \(\theta_x: G \to X\) be defined by \(\theta_x(g) = gx\).

If x ∈ X, then the differential of \(\theta_x\) at the identity maps L into \(X_x\) (the tangent space of X at x). If a ∈ L, we let \(d\theta_x(a)\). It is easy to see that \(\hat{a}\) is a smooth vector field on X. Use \(\tau(X)\) to denote the tangent bundle of X.

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We define a left action of $G^*$ on $\tau(X)$ by

\[(a, g) \cdot v = d\sigma_g(v) + \tilde{a}(g \cdot \pi(v)) \quad \text{for } v \in \tau(X).\]

Here $\pi$ denotes the natural projection from $\tau(X)$ onto $X$ (i.e. $\pi(v) = x$ if and only if $v \in X_x$) and $\sigma_g : X \to X$ is the map $x \mapsto gx$. Clearly, both $d\sigma_g(v)$ and $\tilde{a}(g \cdot \pi(v))$ belong to $X_g.\pi(v)$, so that the sum is defined. We omit the trivial verification that (2) satisfies

\[(0, e) \cdot v = v \quad \text{for all } v \in \tau(X)\]

and

\[((a, g) \cdot (a', g')) \cdot v = (a, g) \cdot ((a', g') \cdot v).\]

Also, it is clear that the action of $G^*$ on $\tau(X)$ defined by (2) is differentiable.

Now assume that $G$ acts transitively and with maximal rank on $X$. If $v$ and $v'$ belong to $\tau(X)$, then there exists $g \in G$ such that $g \cdot \pi(v) = \pi(v')$ (by the transitivity). Since $G$ acts with maximal rank, there is an $a \in L$ such that $d\theta_{\pi(v')}(a) = v' - d\sigma_g(v)$.

Therefore $(a, g) \cdot v = v'$. This shows that $G^*$ acts transitively on $\tau(X)$. We now show that $G^*$ acts with maximal rank. Let $G_0$ be the connected component of the identity element of $G$. Then $G_0$ acts with maximal rank on $X$. Therefore the $G_0$-orbits are open submanifolds of $X$. If $Y$ is a $G_0$-orbit, then $G_0$ acts transitively and with maximal rank on $Y$. We have already shown that this implies that $G_0^*$ acts transitively on $\tau(Y)$. Since $G_0^*$ is obviously connected, it follows that $G_0^*$ acts with maximal rank on $\tau(Y)$. Now $\tau(X)$ is obviously the union of the sets $\tau(Y)$, where $Y$ is a $G_0$-orbit in $X$. These sets are open submanifolds of $\tau(X)$. It follows that $G_0^*$ acts with maximal rank on $\tau(X)$. Then, necessarily, $G^*$ also acts with maximal rank on $\tau(X)$.

REMARKS. (A) If $G$ acts transitively on $X$ it does not follow that $G^*$ acts transitively on $\tau(X)$ (let $X$ = the real line with its usual one-dimensional differentiable structure and $G$—the real line considered as a discrete group).

(B) If $H$ is a closed subgroup of $G$, then $H^*$ can be identified, in an obvious way, with a closed subgroup of $G^*$. One verifies easily that the isotropy group of $0(\varepsilon X_x)$ corresponding to the action of $G^*$ on $\tau(X)$ is precisely $H_x^*$, where $H_x$ is the isotropy group of $x$ corresponding to the action of $G$ on $X$. In particular, we have the diffeomorphism $\tau(G/H) \simeq G^*/H^*$. 

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