PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 2, October 1972

TANGENT BUNDLES OF HOMOGENEOUS SPACES ARE HOMOGENEOUS SPACES

R. W. BROCKETT AND H. J. SUSSMANN

ABSTRACT. In this paper we describe how the tangent bundle of a homogeneous space can be viewed as a homogeneous space.

The purpose of this note is to establish a simple result on the structure of the tangent bundle of a homogeneous space. Even though it is both natural and elementary it does not appear to be in the literature.

We shall associate with every Lie group G another Lie group G^*, constructed as a semidirect product of G with the Lie algebra of G (the precise definition is given below).

Our result is:

THEOREM. *If a Lie group G acts transitively and with maximal rank on a differentiable manifold X, then G^* acts transitively and with maximal rank on the tangent bundle of X.*

Clearly, our result implies that the tangent bundle of a coset space G/H is again a coset space and moreover, is of the form G^*/K for some closed subgroup K of G^*. We will compute K below.

We now define G^* and prove the theorem. Let L be the Lie algebra of G, thought of as the tangent space of G at the identity. For each $g \in G$, we let $ad(g)$ denote the differential at the identity of the inner automorphism $x \to gxg^{-1}$ of G. Thus ad is a (not necessarily one-to-one) homomorphism of G into the group of linear automorphisms of L. We define G^* as the product manifold $L \times G$, with the group operation given by

$$(a, g) \cdot (a', g') = (a + ad(g)(a'), gg'). \tag{1}$$

The verification that G^* is a group is trivial and will be omitted. Also, it is clear that the operation defined by (1) is differentiable, so that G^* is a Lie group.

Now, let G act differentiably on a manifold X. For each $x \in X$, let $\theta_x : G \to X$ be defined by $\theta_x(g) = gx$.

If $x \in X$, then the differential of θ_x at the identity maps L into X_x (the tangent space of X at x). If $a \in L$, we let $\bar{a}(x) = d\theta_x(a)$. It is easy to see that $\bar{a}$ is a smooth vector field on X. Use $\tau(X)$ to denote the tangent bundle of X.

Received by the editors November 24, 1971.
AMS 1970 *subject classifications*. Primary 53C30, 57E25.
Key words and phrases. Tangent bundles, homogeneous spaces.

We define a left action of G^* on $\tau(X)$ by

$$(2) \qquad (a, g) \cdot v = d\sigma_g(v) + \bar{a}(g \cdot \pi(v)) \quad \text{for } v \in \tau(X).$$

Here π denotes the natural projection from $\tau(X)$ onto X (i.e. $\pi(v)=x$ if and only if $v \in X_x$) and $\sigma_g: X \to X$ is the map $x \to gx$. Clearly, both $d\sigma_g(v)$ and $\bar{a}(g \cdot \pi(v))$ belong to $X_{g \cdot \pi(v)}$, so that the sum is defined. We omit the trivial verification that (2) satisfies

$$(0, e) \cdot v = v \quad \text{for all } v \in \tau(X)$$

and

$$((a, g) \cdot (a', g')) \cdot v = (a, g) \cdot ((a', g') \cdot v).$$

Also, it is clear that the action of G^* on $\tau(X)$ defined by (2) is differentiable.

Now assume that G acts transitively and with maximal rank on X. If v and v' belong to $\tau(X)$, then there exists $g \in G$ such that $g \cdot \pi(v) = \pi(v')$ (by the transitivity). Since G acts with maximal rank, there is an $a \in L$ such that $d\theta_{\pi(v')}(a) = v' - d\sigma_g(v)$.

Therefore $(a, g) \cdot v = v'$. This shows that G^* acts transitively on $\tau(X)$. We now show that G^* acts with maximal rank. Let G_0 be the connected component of the identity element of G. Then G_0 acts with maximal rank on X. Therefore the G_0-orbits are open submanifolds of X. If Y is a G_0-orbit, then G_0 acts transitively and with maximal rank on Y. We have already shown that this implies that G_0^* acts transitively on $\tau(Y)$. Since G_0^* is obviously connected, it follows that G_0^* acts with maximal rank on $\tau(Y)$. Now $\tau(X)$ is obviously the union of the sets $\tau(Y)$, where Y is a G_0-orbit in X. These sets are open submanifolds of $\tau(X)$. It follows that G_0^* acts with maximal rank on $\tau(X)$. Then, necessarily, G^* also acts with maximal rank on $\tau(X)$.

REMARKS. (A) If G acts transitively on X it does not follow that G^* acts transitively on $\tau(X)$ (let X= the real line with its usual one-dimensional differentiable structure and G—the real line considered as a discrete group).

(B) If H is a closed subgroup of G, then H^* can be identified, in an obvious way, with a closed subgroup of G^*. One verifies easily that the isotropy group of $0(\varepsilon X_x)$ corresponding to the action of G^* on $\tau(X)$ is precisely H_x^*, where H_x is the isotropy group of x corresponding to the action of G on X. In particular, we have the diffeomorphism $\tau(G/H) \simeq G^*/H^*$.

AIKEN COMPUTATION LABORATORY, HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS 02138 (Current address of R. W. Brockett)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

Current address (H. J. Sussmann): Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903