HENSELIAN FIELDS AND SOLID $k$-VARIETIES. II
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Abstract. Let $k$ be a real closed or Henselian field. A $k$-variety $X$ (affine) is said to be solid if $X$ is determined by its $k$-points. It is shown that a $k$-variety is solid if and only if it contains a nonsingular $k$-point. Another condition for solidity is given and a dimension theorem indicated.

0. Introduction. In [4] a solid $k$-variety is defined to be an affine $k$-variety which is determined by its $k$-points. For real closed and Henselian fields with absolute value, we gave a natural condition which is necessary and sufficient for a $k$-variety to be solid. This condition is here extended to any Henselian field. Moreover, we also demonstrate that for any real closed or Henselian field $k$, a necessary and sufficient condition for a $k$-variety to be solid is that the variety contain a nonsingular $k$ point. This condition is obviously insufficient when dealing with other fields such as the rationals. For example consider the curve $x^3+y^3=1$ defined over the rationals. It has only finitely many rational points all of which are nonsingular.

1. The projection condition. Let $A$ be a local integral domain with maximal ideal $m$. Let $A$, $m$ be Henselian and let $k$ be the quotient field of $A$. Then we wish to show that if $f(x)$ in $k[x]$ has a root in $k$, then so do certain nearby polynomials.

Lemma 1.1 (like [2, Lemma 5.10]). If $A$, $m$ is Henselian, $f(x) \in A[x]$, $\alpha \in A$ so that

1. $f'(\alpha) = \mu$;
2. for some $\delta \in m, f(\alpha) \equiv 0 \mod \mu^2 \delta$;
then $f(x)$ has a root $\alpha' \equiv \alpha \mod \mu m$.

Proof [2]. Let $z$ be a new variable and try to solve $f(x+\mu z) = 0$ for $z$. First expand $f(x+\mu z) = f(x) + \mu f'(x) + \mu^2 w(z)$ where $w(z) \in A[z]$ and is of degree $\geq 2$. Also $f(x) = \mu^2 \delta' \text{ for some } \delta' \in A$. We want to solve
Lemma 1.2. Let $h(x) \in A[x]$ be monic of degree $r$ and suppose

1. $h(\beta) = 0$, $\beta \in A$;
2. $h'(\beta) = \mu \neq 0$.

Then for $\delta \in m$, $g(x) \in A[x]$, $f(x) = h(x) + \mu^2 \delta g(x)$ has a root in $A$.

Proof. Note $f'(\beta) = h'(\beta) + \mu^2 \delta g'(\beta) = \mu u$ for some unit $u \in A$. But $f(\beta) = h(\beta) + \mu^2 \delta g(\beta) = \mu^2 \delta g(\beta)$ and is thus divisible by $\mu^2 \delta$. Now apply Lemma 1.1.

Lemma 1.3. Let $f(x) \in k[x]$ be of degree $r$ and let $f(x)$ have a simple root $\alpha$ in $k$. Then there exists $\gamma \in A$ so that for all $g(x) \in A[x]$ of degree $r$, $f(x) + \gamma g(x)$ has a root in $k$.

Proof. There exists $b \in A$ so that $b \alpha = \beta \in A$. Next define $h(x)$ so that $h(bx) = b^r f(x)$. Then one checks that $h(x) \in A[x]$ and $h(\beta) = 0$. Next $h'(\beta) = \mu \neq 0$ since $\alpha$ is a simple root of $f$. By Lemma 1.2, if $\delta \in m$, $h(x) + \mu^2 \delta g(x)$ has a root $\beta' \in A$ for any $g(x) \in A[x]$. But then $h(bx) + \mu^2 \delta (bx)$ has a root $\beta'/b \in k$. So $f(x) + b^{-r} \mu^2 \delta (bx)$ also has a root $\beta'/b \in k$. But this is sufficient for our purpose since if $g(x) \in A[x]$ is of degree $r$, then $f(x) + b^{-r} \mu^2 \delta (b^{-1}x)$ is in $A[x]$. So if $\gamma = \mu^2 \delta$ then $f(x) + b^{-r} \mu^2 \delta (b^{-1}g(b^{-1}x)) = b(x) + \mu^2 \delta g(x) = f(x) + \gamma g(x)$ has a root in $k$.

Definition 1.3. Given $A$ and $k$ as above, a point $P \in k^d$, and $\lambda \in A$, we wish to define a neighborhood of $P$ in $k^d$ which we call a $\lambda$-sphere. Namely, let $n$ be an integer $\geq 0$ and let

$$S_{P,\lambda,n} = \{Q \in k^d | Q = P + \lambda v, \text{ where } v \in (m^n)^{\times d}\}.$$ 

Definition 1.4. Let $k$ be a field, $\bar{k}$ its algebraic closure. Let $X$ be an affine $k$-variety which we consider as a subset of $\bar{k}^n$ for some $n$. Let $X_k = X \cap k^n$. We say $X$ is solid if $I(X_k) = I(X)$ in $k[X_1, \ldots, X_n]$. By $I(X_k)$ we mean all polynomials in $k[X_1, \ldots, X_n]$ which vanish on $X_k$. Thus $X$ is solid if $X$ is determined by its $k$ points.

We wish to give conditions on $X$ which will be necessary and sufficient for $X$ to be solid.

Lemma 1.5. Let $X$ be a $k$-variety of dimension $d$ and let $\pi : X \to k^d$ be a morphism. Then $X$ is solid if $\pi_k(X_k) \supset S_{P,\lambda,n}$ for some $\lambda$-sphere $S_{P,\lambda,n}$.

Proof. It is easy to see that if $f \in k[Y_1, \ldots, Y_d]$ vanishes on $S_{P,\lambda,n}$, then $f \equiv 0$. If $X$ is not solid, then $X_k \subset W$ for some proper subalgebraic set $W$ of $X$. Then dimension $W < d$ which implies dim $\pi(W) < d$. But $\pi(W) \supset S_{P,\lambda,n}$ implies dimension $\pi(W) \geq d$, a contradiction.
Let $X$ be a variety of dimension $d$. Then if $k[x_1, \ldots, x_n]$ is the coordinate ring of $X$, by Noether normalization [6, p. 266], we can assume $x_1, \ldots, x_d$ are independent transcendentals and $k[x_1, \ldots, x_n]$ is integral and separable over $k[x_1, \ldots, x_d]$. Let $\pi: X \to \bar{k}^d$ be the induced morphism.

**Proposition 1.6.** Let $k$ be Henselian, i.e., the quotient field of a Henselian ring $A$. Let $X$ be a $k$-variety of dimension $d$ and $\pi: X \to \bar{k}^d$ as above. Then $X$ is solid if and only if $\pi(X_k)$ contains a $\lambda$-sphere.

**Proof.** The proof is the same as that given in [4] except that here one gets a $\lambda$-sphere. First choose $z \in k[x_1, \ldots, x_n]$ so that the quotient field of $k[x_1, \ldots, x_d, z]=\text{quotient field of } k[x_1, \ldots, x_n]$. Then let $f(x_1, \ldots, x_d, Z)=\text{the primitive irreducible polynomial of } z$ over $k[x_1, \ldots, x_d]$. Then $f(x_1, \ldots, x_d, Z)=\sum_{i=0}^m a_i(x_1, \ldots, x_d)Z^i$ is irreducible in $k[x_1, \ldots, x_d, Z]$.

Now $x_{d+1} z=\sum_{j=0}^m (b_{ij}(x_1, \ldots, x_d))c_{ij}(x_1, \ldots, x_d)z^j$ where $b_{ij}, c_{ij} \in k[x_1, \ldots, x_d]$. We let $U=\{P' \in X|a_m(P') \neq 0, \text{ all } c_{ij}(P') \neq 0 \text{ and } (\partial f/\partial z)(P') \neq 0\}$. Noting $U$ is nonempty, we can choose $P' \in U$. Let $P=\pi(P')$.

Now choose $\lambda$ so that if $Q \in S_{P, \lambda, 1}$ then $a_m(Q) \neq 0$, all $c_{ij}(Q) \neq 0$ and, using Lemma 1.3, $f(Q, Z)$ has a root $\alpha \in k$.

Then we let $Q'=(x_1(Q), \ldots, x_d(Q), x_{d+1}(Q, \alpha), \ldots, x_n(Q, \alpha))$. And as in [4] show that $Q'$ is a $k$ point of $X$ and $\pi(Q')=Q$. This shows $\pi(X_k) \supset S_{P, \lambda, 1}$.

From Proposition 1.6 it is possible to prove as in [4] a dimension theorem.

**Theorem 1.7.** Let $k$ be a Henselian field, $X$ a solid $k$-variety of dimension $d$. Let $W_1, \ldots, W_r$ be subvarieties of $X$ of dimension $\leq d-2$. Then there exists a solid $k$-variety $W$ contained in $X$ with dimension $W=d-1$ and $W \supset W_1 \cup \cdots \cup W_r$.

**Proof.** Just as in [4, Theorem 3].

2. The nonsingular point condition.

**Theorem 2.1.** Let $k$ be Henselian or real closed. A $k$-variety $X$ is solid if and only if $X_k$ contains a nonsingular point of $X$.

**Proof.** First the Henselian case. Let $Q$ be a nonsingular point of $X$. Let $k=$ the quotient field of $A$ and let $k[x_1, \ldots, x_n]$ be the polynomial ring. Let $d=\dim X$. Then we can find $f_1, \ldots, f_r, r=n-d$, in $k[x_1, \ldots, x_n]$ so that $X=V(f_1, \ldots, f_r)$ in a neighborhood of $Q$. We can assume all $f_i \in A[x_1, \ldots, x_n]$. We know rank $((\partial f_i/\partial x_j)(Q))=r$. Then by reordering the $x_i$'s, we can assume

$$\det_{i,j=1,\ldots,r} ((\partial f_i/\partial x_j)(Q)) = \mu \neq 0.$$
We next want to apply Lemma 5.10 of [2]. To do this, we need to change \( Q=(a_1, \ldots, a_r, b_1, \ldots, b_d)=(a, b) \). There exists \( \gamma \in A \) so that \( \gamma a_i, \gamma b_j \in A \) for all \( i, j \). Let \( \gamma a=(\gamma a_1, \ldots, \gamma a_r) \). Let \( d_i=\)degree of \( f_i \). We then let \( h_i(\gamma x)=\gamma^d f_i(x) \) and then \( h_i(\gamma a, \gamma b)=0 \). Moreover \( (\partial h_i/\partial x_j)(\gamma a, \gamma b)=\gamma^{d_i-1}(\partial f_i/\partial x_j)(a, b) \) so we get \( (\partial h_{i,j}=1, \ldots, r)(\partial h_i/\partial x_j)(\gamma a, \gamma b)=\gamma^s \mu \) for some \( s \). Now choose \( b' \) so \( \gamma b'=\gamma b+\gamma^{2s}x^2 \) where \( v \in m^{I=d} \). Then \( h_i(\gamma a, \gamma b')=0 \mod \mu^2 \gamma \). We then let \( \gamma^s \mu \) mod \( \mu^2 \gamma \), and so

\[
\det((\partial h_j/\partial x_j)(\gamma a, \gamma b'))=\gamma^s \mu \quad \text{mod} \quad \mu^2 \gamma,
\]

and so \( \gamma^s \mu \) where \( u \) is a unit in \( A \).

By Lemma 5.10 of [2], there exists \( a' \in k \) so that \( h_i(\gamma a', \gamma b')=0 \) for all \( i \). Then \( f_i(a', b')=0 \) for all \( i \). This means \( b' \in \pi(X_k) \) where \( \pi(a, b)=b \). Letting \( \lambda=\mu^2 \gamma \) and \( P=\pi(Q) \), we have \( \pi(X_k) \supset S_{P,1,1} \), and so by Lemma 1.5, \( X \) is solid.

For the real closed case, we need to prove an implicit function theorem.

**Lemma 2.2.** Let \( k \) be a real closed field and \( f_1, \ldots, f_r \in k[x_1, \ldots, x_r] \). Let \( X=V(f_1, \ldots, f_r) \). Let \( P \in X_k \) and suppose \( \det((\partial f_i/\partial x_j)(P)\neq0, i, j=1, \ldots, r \). Then there exists \( \epsilon \neq0 \) in \( k \) such that the following holds: Let \( P=(a_1, \ldots, a_r) \), then if \( \sum_{i=r+1}^n (b_i-a_i)^2 \leq \epsilon^2 \), there exist \( b_1, \ldots, b_r \) in \( k \) such that \( f_i(b_1, \ldots, b_r)=0, i=1, \ldots, r \). In other words, \( \sum_{i=r+1}^n (b_i-a_i)^2 \leq \epsilon^2 \) implies \( (b_{r+1}, \ldots, b_n) \) in \( \pi(X_k) \) where \( \pi:X \to k^d \) is the obvious projection.

**Proof.** We apply the Tarski-Seidenberg criterion given in Jacobson
[5, p. 314], which states:

Let \( t=(t_1, \ldots, t_r) \), \( x=(x_1, \ldots, x_n) \). Let \( f \in Q[t, x] \), \( Q \) the rational numbers. Then let \( f(t, x)=0 \) be an equality which has solutions for \( x \) in \( k^n \) for all substitutions for \( t \) in \( k^r \), for some real closed field \( k \).

**Conclusion.** For every real closed field \( k \), we have solutions for \( x \) in \( k^n \) for all substitutions for \( t \) in \( k^r \).

To translate our situation to the above, we must add new variables
\( t=(t_1, \ldots, t_r) \) as "dummy variables" to get polynomials \( f(t, x) \in Q[t, x] \) so that substituting correctly for \( t \) in \( k^r \), we obtain the \( f_i(x) \). Next let

\[
g(t, x_1, \ldots, x_n)=\det((\partial f_i/\partial x_j)(t, x), i, j=1, \ldots, r.\) Adding a new variable \( x_{n+1} \), we consider the polynomial

\[
f(t, x_1, \ldots, x_n, x_{n+1})=\sum_{i=1}^r f_i^2(t, x) + (1 - x_{n+1} g(t, x))^2 \in Q[t, x].
\]

Add new variables \( y_{r+1}, \ldots, y_n, z, \epsilon \) and let

\[
h(x_{r+1}, \ldots, x_n, y_{r+1}, \ldots, y_n, z, \epsilon)=\sum_{i=r+1}^n (x_i - y_i)^2 - \epsilon^2 + z^2.
\]

Now note that the statement in Lemma 2.2 is equivalent to: For every substitution of \( t_0 \) for \( t \) in \( k^r \), if the equation \( f(t_0, x_1, \ldots, x_{n+1})=0 \) has a
solution for \( x_1, \ldots, x_{n+1} \) in \( k \), then there exists \( \epsilon \neq 0 \) so that if \( y_{r+1}, \ldots, y_n, z, \epsilon \) \in \( k \) and \( h(x_{r+1}, \ldots, x_n, y_{r+1}, \ldots, y_n, z, \epsilon) = 0 \), there exists \( y_1, \ldots, y_r \in k \) so that \( f(t_0, y) = 0 \). Add new variables \( u, v \) and let

\[
\alpha(t, x, y, z, u, v, \epsilon) = (1 - uf(t, x))^2(1 - ev)^2 + (1 - h(x, y, z, \epsilon)w)^2f^2(t, y).
\]

Now one verifies that Lemma 2.2 for \( k \) is equivalent to the statement that \( \alpha(t, x, y, z, u, v, \epsilon) = 0 \) has a solution in the remaining variables for all choices of \( t_1, \ldots, t_s, x_1, \ldots, x_{n+1}, y_{r+1}, \ldots, y_n, z \) in \( k \). But Lemma 2.2 is true for \( k = \mathbb{R} \), the real numbers, by the implicit function theorem \([1, \text{p. 147}]\). Thus, by the theorem in Jacobson \([5]\) quoted at the start of the proof, Lemma 2.2 is true for all real closed fields \( k \).

To apply Lemma 2.2 to prove the real part of Theorem 2.1, choose \( P \) nonsingular in \( X_k \). As in the Henselian case, we can find a neighborhood of \( P \) where \( X = V(f_1, \ldots, f_r), f_i \in k[x_1, \ldots, x_n] \), \( r = n - d \), \( d = \dim X \) and

\[
\det_{i,j=1,\ldots,r}(\partial f_i/\partial x_j)(P) \neq 0.
\]

By Lemma 2.2, \( \pi(X_k) \) contains a sphere in \( k^d \). Then by the real equivalent of Lemma 1.5 (see \([3, \text{Theorem 1}]\) we are done.

To see that a solid \( k \)-variety contains a nonsingular \( k \) point, just note that the set \( U \) of nonsingular points of \( X \) is Zariski open \( X \); and, since \( X \) is solid, \( X_k \cap U \) is not empty.

Bibliography


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