PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 2, October 1972

HENSELIAN FIELDS AND SOLID k-VARIETIES. II

GUSTAVE EFROYMSON

ABSTRACT. Let k be a real closed or Henselian field. A k-variety X (affine) is said to be solid if X is determined by its k points. It is shown that a k-variety is solid if and only if it contains a nonsingular k point. Another condition for solidity is given and a dimension theorem indicated.

0. **Introduction.** In [4] a solid k-variety is defined to be an affine k-variety which is determined by its k points. For real closed and Henselian fields with absolute value, we gave a natural condition which is necessary and sufficient for a k-variety to be solid. This condition is here extended to any Henselian field. Moreover, we also demonstrate that for any real closed or Henselian field k, a necessary and sufficient condition for a k-variety to be solid is that the variety contain a nonsingular k point. This condition is obviously insufficient when dealing with other fields such as the rationals. For example consider the curve $x^3+y^3=1$ defined over the rationals. It has only finitely many rational points all of which are nonsingular.

1. **The projection condition.** Let A be a local integral domain with maximal ideal m. Let A, m be Henselian and let k be the quotient field of A. Then we wish to show that if $f(x)$ in $k[x]$ has a root in k, then so do certain nearby polynomials.

LEMMA 1.1 (*like* [**2**, *Lemma* 5.10]). *If A, m is Henselian, $f(x)\in A[x]$, $\alpha\in A$ so that*

(1) $f'(\alpha)=\mu$;

(2) *for some* $\delta\in m$, $f(\alpha)\equiv 0 \bmod \mu^2\delta$;

then $f(x)$ has a root $\alpha'\equiv\alpha \bmod \mu m$.

PROOF [**2**]. Let z be a new variable and try to solve $f(\alpha+\mu z)=0$ for z. First expand $f(\alpha+\mu z)=f(\alpha)+\mu z f'(\alpha)+\mu^2 w(z)$ where $w(z)\in A[z]$ and is of degree $\geqq 2$. Also $f(\alpha)=\mu^2\delta\delta'$ for some $\delta'\in A$. We want to solve

Received by the editors September 14, 1971.

AMS 1970 *subject classifications*. Primary 14G25; Secondary 12D15, 13J15.

Key words and phrases. Solid k-variety, Henselian field, real closed field, implicit function theorem.

$\mu^2\delta\delta'+\mu^2z+\mu^2w(z)=0$ for z. Let $h(z)=\delta\delta'+z+w(z)$. Then $h(0)=\delta\delta'\in m$, $h'(0)=1$. But A is Henselian so h has a root $z_0\in m$. Then $f(\alpha+\mu z_0)=0$.

LEMMA 1.2. *Let $h(x)\in A[x]$ be monic of degree r and suppose*
(1) $h(\beta)=0$, $\beta\in A$;
(2) $h'(\beta)=\mu\neq 0$.
Then for $\delta\in m$, $g(x)\in A[x]$, $f(x)=h(x)+\mu^2\delta g(x)$ has a root in A.

PROOF. Note $f'(\beta)=h'(\beta)+\mu^2\delta g'(\beta)=\mu u$ for some unit $u\in A$. But $f(\beta)=h(\beta)+\mu^2\delta g(\beta)=\mu^2\delta g(\beta)$ and is thus divisible by $\mu^2\delta$. Now apply Lemma 1.1.

LEMMA 1.3. *Let $f(x)\in k[x]$ be of degree r and let $f(x)$ have a simple root α in k. Then there exists $\gamma\in A$ so that for all $g(x)$ in $A[x]$ of degree r, $f(x)+\gamma g(x)$ has a root in k.*

PROOF. There exists $b\in A$ so that $b\alpha=\beta\in A$. Next define $h(x)$ so that $h(bx)=b^rf(x)$. Then one checks that $h(x)\in A[x]$ and $h(\beta)=0$. Next $h'(\beta)=\mu\neq 0$ since α is a simple root of f. By Lemma 1.2, if $\delta\in m$, $h(x)+\mu^2\delta g(x)$ has a root $\beta'\in A$ for any $g(x)\in A[x]$. But then $h(bx)+\mu^2\delta g(bx)$ has a root $\beta'/b\in k$. So $f(x)+b^{-r}\mu^2\delta g(bx)$ also has a root $\beta'/b\in k$. But this is sufficient for our purpose since if $g(x)\in A[x]$ is of degree r, then $f^rg(b^{-1}x)$ is in $A[x]$. So if $\gamma=\mu^2\delta$ then $f(x)+b^{-r}\mu^2\delta(b^rg(b\cdot b^{-1}x))=b(x)+\mu^2\delta g(x)=f(x)+\gamma g(x)$ has a root in k.

DEFINITION 1.3. Given A and k as above, a point $P\in k^d$, and $\lambda\in A$, we wish to define a neighborhood of P in k^d which we call a λ-sphere. Namely, let n be an integer $\geqq 0$ and let

$$S_{P,\lambda,n}=\{Q\in k^d \mid Q=P+\lambda v, \text{ where } v\in(m^n)^{\times d}\}.$$

DEFINITION 1.4. Let k be a field, $\bar{k}$ its algebraic closure. Let X be an affine k-variety which we consider as a subset of $\bar{k}^n$ for some n. Let $X_k=X\cap k^n$. We say X is *solid* if $I(X_k)=I(X)$ in $k[X_1,\cdots,X_n]$. By $I(X_k)$ we mean all polynomials in $k[X_1,\cdots,X_n]$ which vanish on X_k. Thus X is solid if X is determined by its k points.

We wish to give conditions on X which will be necessary and sufficient for X to be solid.

LEMMA 1.5. *Let X be a k-variety of dimension d and let $\pi: X\to\bar{k}^d$ be a morphism. Then X is solid if $\pi_k(X_k)\supset S_{P,\lambda,n}$ for some λ-sphere $S_{P,\lambda,n}$.*

PROOF. It is easy to see that if $f\in k[Y_1,\cdots,Y_d]$ vanishes on $S_{P,\lambda,n}$, then $f\equiv 0$. If X is not solid, then $X_k\subset W$ for some proper subalgebraic set W of X. Then dimension $W<d$ which implies $\dim\pi(W)<d$. But $\pi(W)\supset S_{P,\lambda,n}$ implies dimension $\pi(W)\geqq d$, a contradiction.

Let X be a variety of dimension d. Then if $k[x_1, \cdots, x_n]$ is the coordinate ring of X, by Noether normalization [**6**, p. 266], we can assume $x_1, \cdots, x_d$ are independent transcendentals and $k[x_1, \cdots, x_n]$ is integral and separable over $k[x_1, \cdots, x_d]$. Let $\pi: X \to \bar{k}^d$ be the induced morphism.

PROPOSITION 1.6. *Let k be Henselian, i.e., the quotient field of a Henselian ring A. Let X be a k-variety of dimension d and $\pi: X \to \bar{k}^d$ as above. Then X is solid if and only if $\pi(X_k)$ contains a λ-sphere.*

PROOF. The proof is the same as that given in [**4**] except that here one gets a λ-sphere. First choose $z \in k[x_1, \cdots, x_n]$ so that the quotient field of $k[x_1, \cdots, x_d, z]$=quotient field of $k[x_1, \cdots, x_n]$. Then let $f(x_1, \cdots, x_d, Z)$=the primitive irreducible polynomial of z over $k(x_1, \cdots, x_d)$. Then $f(x_1, \cdots, x_d, Z) = \sum_{i=0}^{m} a_i(x_1, \cdots, x_d)Z^i$ is irreducible in $k[x_1, \cdots, x_d, Z]$.

Now $x_{d+i} = \sum_{j=0}^{m-1} (b_{ij}(x_1, \cdots, x_d)/c_{ij}(x_1, \cdots, x_d))z^j$ where $b_{ij}, c_{ij} \in k[x_1, \cdots, x_d]$. We let $U = \{P' \in X \mid a_m(P') \neq 0,$ all $c_{ij}(P') \neq 0$ and $(\partial f/\partial z)(P') \neq 0\}$. Noting U is nonempty, we can choose $P' \in U$. Let $P = \pi(P')$.

Now choose λ so that if $Q \in S_{P,\lambda,1}$ then $a_m(Q) \neq 0$, all $c_{ij}(Q) \neq 0$ and, using Lemma 1.3, $f(Q, Z)$ has a root $\alpha \in k$.

Then we let $Q' = (x_1(Q), \cdots, x_d(Q), x_{d+1}(Q, \alpha), \cdots, x_n(Q, \alpha))$. And as in [**4**] show that Q' is a k point of X and $\pi(Q') = Q$. This shows $\pi(X_k) \supset S_{P,\lambda,1}$.

From Proposition 1.6 it is possible to prove as in [**4**] a dimension theorem.

THEOREM 1.7. *Let k be a Henselian field, X a solid k-variety of dimension d. Let $W_1, \cdots, W_r$ be subvarieties of X of dimension $\leqq d-2$. Then there exists a solid k-variety W contained in X with dimension $W = d-1$ and $W \supset W_1 \cup \cdots \cup W_r$.*

PROOF. Just as in [**4**, Theorem 3].

2. The nonsingular point condition.

THEOREM 2.1. *Let k be Henselian or real closed. A k-variety X is solid if and only if X_k contains a nonsingular point of X.*

PROOF. First the Henselian case. Let Q be a nonsingular point of X. Let k=the quotient field of A and let $k[x_1, \cdots, x_n]$ be the polynomial ring. Let d=dim X. Then we can find $f_1, \cdots, f_r$, $r = n-d$, in $k[x_1, \cdots, x_n]$ so that $X = V(f_1, \cdots, f_r)$ in a neighborhood of Q. We can assume all $f_i \in A[x_1, \cdots, x_n]$. We know $\operatorname{rank}((\partial f_i/\partial x_j)(Q)) = r$. Then by reordering the x_i's, we can assume

$$\det_{i,j=1,\cdots,r} ((\partial f_i/\partial x_j)(Q)) = \mu \neq 0.$$

We next want to apply Lemma 5.10 of [2]. To do this, we need to change $Q=(a_1, \cdots, a_r, b_1, \cdots, b_d)=(a, b)$. There exists $\gamma \in A$ so that γa_i, $\gamma b_j \in A$ for all i, j. Let $\gamma a=(\gamma a_1, \cdots, \gamma a_r)$. Let d_i=degree of f_i. We then let $h_i(\gamma x)=\gamma^{d_i} f_i(x)$ and then $h_i(\gamma a, \gamma b)=0$. Moreover $(\partial h_i/\partial x_j)(\gamma a, \gamma b)=\gamma^{d_i-1}(\partial f_i/\partial x_j)(a, b)$ so we get $\det_{i,j=1,\cdots,r}(\partial h_i/\partial x_j)(\gamma a, \gamma b)=\gamma^s\mu$ for some s. Now choose b' so $\gamma b'=\gamma b+\gamma^{2s}\mu^2 v$ where $v \in m^{\times d}$. Then $h_i(\gamma a, \gamma b')\equiv 0 \bmod \mu^2\gamma^{2s}m$, all i. And

$$\begin{aligned}\det((\partial h_i/\partial x_j)(\gamma a, \gamma b')) &\equiv \gamma^s\mu \bmod \gamma^{2s}\mu^2 m, \quad \text{and so}\\ &= \gamma^s \mu u \qquad\qquad \text{where } u \text{ is a unit in } A.\end{aligned}$$

By Lemma 5.10 of [2], there exists $a' \in k$ so that $h_i(\gamma a', \gamma b')=0$ for all i. Then $f_i(a', b')=0$ for all i. This means $b' \in \pi(X_k)$ where $\pi(a, b)=b$. Letting $\lambda=\mu^2\gamma^{2s}$ and $P=\pi(Q)$, we have $\pi(X_k)\supset S_{P,\lambda,1}$, and so by Lemma 1.5, X is solid.

For the real closed case, we need to prove an implicit function theorem.

LEMMA 2.2. *Let k be a real closed field and $f_1, \cdots, f_r \in k[x_1, \cdots, x_n]$. Let $X=V(f_1, \cdots, f_r)$. Let $P \in X_k$ and suppose* $\det(\partial f_i/\partial x_j)(P)\neq 0$, $i, j=1, \cdots, r$. *Then there exists $\varepsilon\neq 0$ in k such that the following holds: Let $P=(a_1, \cdots, a_n)$, then if $\sum_{i=r+1}^n (b_i-a_i)^2\leqq\varepsilon^2$, there exist $b_1, \cdots, b_r$ in k such that $f_i(b_i, \cdots, b_n)=0$, $i=1, \cdots, r$. In other words, $\sum_{i=r+1}^n (b_i-a_i)^2\leqq \varepsilon^2$ implies $(b_{r+1}, \cdots, b_n)$ in $\pi(X_k)$ where $\pi: X\to\bar{k}^d$ is the obvious projection.*

PROOF. We apply the Tarski-Seidenberg criterion given in Jacobson [5, p. 314], which states:

Let $t=(t_1, \cdots, t_r)$, $x=(x_1, \cdots, x_n)$. Let $f \in Q[t, x]$, Q the rational numbers. Then let $f(t, x)=0$ be an equality which has solutions for x in k^n for all substitutions for t in k^r, for some real closed field k.

Conclusion. For every real closed field k, we have solutions for x in k^n for all substitutions for t in k^r.

To translate our situation to the above, we must add new variables $t=(t_1, \cdots, t_s)$ as "dummy variables" to get polynomials $f(t, x) \in Q[t, x]$ so that substituting correctly for t in k^s, we obtain the $f_i(x)$. Next let $g(t, x_1, \cdots, x_n)=\det(\partial f_i/\partial x_j)$, $i, j=1, \cdots, r$. Adding a new variable x_{n+1}, we consider the polynomial

$$f(t, x_1, \cdots, x_{n+1}) = \sum_{i=1}^{r} f_i^2(t, x) + (1 - x_{n+1}g(t, x))^2 \in Q[t, x].$$

Add new variables $y_{r+1}, \cdots, y_n, z, \varepsilon$ and let

$$h(x_{r+1}, \cdots, x_n, y_{r+1}, \cdots, y_n, z, \varepsilon) = \sum_{i=r+1}^{n} (x_i - y_i)^2 - \varepsilon^2 + z^2.$$

Now note that the statement in Lemma 2.2 is equivalent to: For every substitution of t_0 for t in k^s, if the equation $f(t_0, x_1, \cdots, x_{n+1})=0$ has a solution for $x_1, \cdots, x_{n+1}$ in k, then there exists $\varepsilon \neq 0$ so that if $y_{r+1}, \cdots, y_n, z \in k$ and $h(x_{r+1}, \cdots, x_n, y_{r+1}, \cdots, y_n, z, \varepsilon)=0$, there exists $y_1, \cdots, y_r \in k$ so that $f(t_0, y)=0$. Add new variables u, v and let

$$\alpha(t,x,y,z,u,v,\varepsilon) = (1 - uf(t,x))^2(1 - \varepsilon v)^2 + (1 - h(x,y,z,\varepsilon)w)^2 f^2(t,y).$$

Now one verifies that Lemma 2.2 for k is equivalent to the statement that $\alpha(t, x, y, z, u, v, \varepsilon)=0$ has a solution in the remaining variables for all choices of $t_1, \cdots, t_s, x_1, \cdots, x_{n+1}, y_{r+1}, \cdots, y_n, z$ in k. But Lemma 2.2 is true for $k=R$, the real numbers, by the implicit function theorem [**1**, p. 147]. Thus, by the theorem in Jacobson [**5**] quoted at the start of the proof, Lemma 2.2 is true for all real closed fields k.

To apply Lemma 2.2 to prove the real part of Theorem 2.1, choose P nonsingular in X_k. As in the Henselian case, we can find a neighborhood of P where $X=V(f_1, \cdots, f_r)$, $f_i \in k[x_1, \cdots, x_n]$, $r=n-d$, $d=\dim X$ and $\det_{i,j=1,\cdots,r}(\partial f_i/\partial x_j)(P)\neq 0$.

By Lemma 2.2, $\pi(X_k)$ contains a sphere in k^d. Then by the real equivalent of Lemma 1.5 (see [**3**, Theorem 1]) we are done.

To see that a solid k-variety contains a nonsingular k point, just note that the set U of nonsingular points of X is Zariski open X; and, since X is solid, $X_k \cap U$ is not empty.

Bibliography

1. T. M. Apostol, *Mathematical analysis: A modern approach to advanced calculus*, Addison-Wesley, Reading, Mass., 1957. MR **19**, 398.

2. M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. No. 36 (1969), 23–58. MR **42** #3087.

3. D. Dubois and G. Efroymson, *A dimension theorem for real primes* (submitted).

4. G. Efroymson, *Henselian fields and solid k varieties*. I, Trans. Amer. Math. Soc. **170** (1972) 187–195.

5. N. Jacobson, *Lectures in abstract algebra*. Vol. III: *Theory of fields and Galois theory*, Van Nostrand, Princeton, N.J., 1964. MR **30** #3087.

6. O. Zariski and P. Samuel, *Commutative algebra*. Vol. 1, University Series in Higher Math., Van Nostrand, Princeton, N.J., 1958. MR **19**, 833.

Department of Mathematics, University of New Mexico, Albuquerque, New Mexico 87106