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QUASI-INVARIANT RADON MEASURES ON GROUPS

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ABSTRACT. Let G be a Hausdorff topological group which is a Baire space. It is proved that if there is a quasi-invariant Radon measure on G then G is locally compact. Examples of non-Baire groups with and without quasi-invariant measures are considered. In particular, it is shown that there is no σ-finite measure on the Wiener space which preserves sets of measure zero under translation.

Let G be a Hausdorff topological group. A Radon measure μ on G is said to be left quasi-invariant if for every $\sigma \in G$ the measures μ and μ_σ (the image of the measure μ under the mapping $x \to \sigma x$) are absolutely continuous relative to each other. This is equivalent to the property that for any compact set $K \subset G$, $\mu(K)=0$ iff $\mu(\sigma^{-1}K)=0$. It is a well-known fact that on a locally compact Hausdorff group the left quasi-invariant Radon (q.-i.R.) measures which are also right quasi-invariant are the indefinite integrals, with respect to the left invariant Haar measure, of a positive measurable function f such that $f>0$ except on a locally (Haar) negligible set. However, in case G is not locally compact, there may exist left q.-i.R. measures, even though there does not exist any left invariant Radon measure [3]. All the groups considered below will be Hausdorff. We prove the following:

THEOREM 1. *Let G be a topological group which is a Baire space. If there is a nontrivial* ($\not\equiv 0$) *left q.-i.R. measure on G then G is locally compact.*

The proof is based on the following lemma and on an idea due to J. C. Oxtoby [5, Theorem 2].

LEMMA 1. *Let $\mu \not\equiv 0$ be a left q.-i.R. measure on a topological group G. Then $\mu(U)>0$ for every nonvoid open set $U \subset G$.*

PROOF. Suppose on the contrary $\mu(U)=0$ for some nonvoid open set U contained in G. Since μ is quasi-invariant it follows that every point has an open neighbourhood of μ measure zero. Hence every compact set has μ-measure zero and this in turn implies that $\mu\equiv 0$; a contradiction. The lemma is proved.

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PROOF OF THE THEOREM. Since μ is locally finite and, by the above lemma, we may choose an open set V containing the identity element e such that $0<\mu(V)<+\infty$, we may also assume that V is symmetric. By the inner regularity of μ we can find a compact set $K\subset V$ such that $\mu(K)>0$ and, if necessary, by replacing K by $K\cup K^{-1}$, which is also contained in V, we may assume that $K=K^{-1}$. Let $G_1=\bigcup_{n=1}^{\infty} K^n$; G_1 is a subgroup of G.

Let us suppose that G is not locally compact. Then every compact subset of G has void interior. In particular, K^n for each n has void interior. Hence G_1 is a meagre set. Let, now, W be an open neighbourhood of e such that $WK\subset V$. Let W' be a set of representatives from W of the cosets of W by G_1. Then $F=\bigcup \sigma G_1$ for $\sigma\in W'$ is a disjoint union of meagre sets and clearly $W\subset F$. Since G is a Baire space so is W and hence W' is necessarily an uncountable set. Now, for any σ, $\sigma'\in W'$, $\sigma K\cap\sigma' K\subset \sigma G_1\cap\sigma' G_1=\varnothing$ and also $\sigma' K\subset V$, $\sigma K\subset V$. But by the choice of K, $\mu(K)>0$ and since μ is left quasi-invariant $\mu(\sigma K)>0$ for every σ. We deduce that there are uncountably many mutually disjoint compact sets (viz. copies of K) of μ-measure >0 contained in V. This implies that $\mu(V)=+\infty$; which contradicts the choice $\mu(V)<+\infty$. Hence G is locally compact, proving the theorem.

We consider below some examples of non-Baire groups relative to the question of the existence of q.-i.R. measures.

EXAMPLE 1. Let G be a countable group and τ a Hausdorff topology which is not the discrete topology. The additive group of rational numbers Q provides one such example. Also, the additive group of integers can be given a nonmetrizable topology (obviously not locally compact) under which it is a Hausdorff topological group [**4**, p. 27]. On any such topological group G the following measure μ is obviously Radon and both left and right quasi-invariant. Suppose $G=\{a_n\}_1^{\infty}$ and let $\mu(E)=\sum_{k\in M}(1/2^k)$ if $E=\{a_k | k\in M\subset N\}$.

EXAMPLE 2. Consider the product group $G\times H$ with the product topology where G is as in Example 1 and H is a locally compact group (and H uncountable). Let $\lambda=\mu\otimes\nu$ where μ ($\not\equiv 0$) is a quasi-invariant measure on G and ν a left invariant Haar measure on H. It is obvious that $G\times H$ is uncountable and not Baire. The Radon measure λ on $G\times H$ is left quasi-invariant in virtue of the following proposition.

PROPOSITION 1. *Let μ_i be a left q.-i.R. measure on a group G_i, $i=1$ to n. Then on the product $G=\prod_{i=1}^{n} G_i$, the Radon measure $\mu=\mu_1\otimes\mu_2\otimes\cdots\otimes\mu_n$ is left quasi-invariant.*

PROOF. It is sufficient to prove the proposition for $n=2$. In the general case the proof follows by induction. Suppose K is a compact set $\subset G_1\times G_2$

such that $\mu(K)=0$. Let K_y denote the section of K through $y \in G_2$, and $A=\{y \in G_2 : \mu_1(K_y)>0\}$. By Fubini's theorem $\mu_2(A)=0$. Suppose $(x_0, y_0) \in G$ and $K'=(x_0, y_0)K$. We observe that $K'_{y_0 y}=x_0 K_y$ and by the left quasi-invariance of μ_1 we get that $\mu_1(K'_{y_0 y})=0$, for every $y \notin A$. Also $\mu_2(y_0 A)=0$. Again by Fubini's theorem, we conclude that $\mu(K')=0$. The proof is complete.

REMARK. The above result does not hold good for infinite products. Consider $G=\boldsymbol{R}^N$ which is a Baire group. The product of quasi-invariant measures of total mass 1 on $\boldsymbol{R}$ is not quasi-invariant on G since G is not locally compact.

EXAMPLE 3. On the other hand, not every non-Baire group has a left or right q.-i.R. measure on it. In [**2**] Feldman, generalizing a result of Sudakov [**7**], has proved that every metrizable linear space with a non-trivial σ-finite quasi-invariant Borel measure is necessarily finite dimensional. From this it is possible to get a wide class of topological vector spaces that do not admit any q.-i.R. measure on them. In particular, the following two results could be deduced as consequences of Feldman's result. However, we give below proofs based on Theorem 1.

THEOREM 2. *Let F be an infinite dimensional separable Fréchet space. Let τ be any weaker Hausdorff topology on F such that (F, τ) is a topological vector space. Then on (F, τ) considered as an Abelian group, there is no q.-i.R. measure.*

PROOF. Suppose μ is a q.-i.R. measure on (F, τ). By the local finiteness of μ and the fact that τ on F is Lindelöf (being weaker than a Lindelöf topology) we deduce that μ is necessarily σ-finite. Replacing μ by an equivalent measure, if necessary, we may assume that μ is totally finite. Now F, a separable Fréchet space, is polish and hence the finite Radon measures (and the Borel sets) are the same for F and (F, τ) [**6**, Part II]. Hence, μ is, in fact, a Radon measure on F and it is quasi-invariant. But F being an infinite dimensional Fréchet space (in particular Baire), this is impossible by Theorem 1. The proof is complete.

REMARK. Suppose F is the strong dual of an infinite dimensional Banach space E and F separable. Then F with weak* topology is σ-compact (viz. the strong closed balls of radii n centre 0 are weak* compact and form a countable covering) and is hence non-Baire. However, by the above theorem there is no quasi-invariant Radon measure on this space.

EXAMPLE 4. The following is a generalization of a theorem of R. H. Cameron [**1**]. Let $\mathscr{C}_0[0, 1]=W$ be the space of real valued continuous functions on $[0, 1]$ vanishing at the origin. Let $\mathscr{U}$ (resp. τ) be the topology of uniform (resp. pointwise) convergence on W. Since $(W, \mathscr{U})$ is polish the Borel σ-algebra $\mathscr{B}$ is the same for (W, τ) and $(W, \mathscr{U})$. This σ-algebra is also the Wiener σ-algebra on W. It is obvious that every set in the Wiener σ-algebra belongs to $\mathscr{B}$. Conversely, a base for open sets in τ (viz. sets of the form $\bigcap_1^k \{f \in W : f(x_i) \in V_i\}$ where $x_1, \cdots, x_k \in [0, 1]$ and V_i open sets of $\boldsymbol{R}$) is contained in the Wiener σ-algebra. However, since (W, τ) is a Lindelöf space, every τ-open set is a countable union of open sets belonging to the above base and hence is in the Wiener σ-algebra. From this it is easily seen that $\mathscr{B}$ is identical with the Wiener σ-algebra.

THEOREM 3. *There is no σ-finite measure on $\mathscr{B}$ on the space W which preserves sets of measure zero under translation.*

PROOF. Suppose there is one such measure μ. We may assume without loss of generality that there is a probability measure ν on $\mathscr{B}$ which preserves sets of measure zero under translation. But then looking at $\mathscr{B}$ as the Borel σ-algebra of $(W, \mathscr{U})$ we conclude that ν is a Radon measure on $(W, \mathscr{U})$, a polish space. This is impossible in view of the Theorem 1. Hence such a measure cannot exist; this completes the proof.

REMARK. Suppose μ is a σ-finite measure on $\mathscr{B}$. Then it follows from the above theorem that $\exists$ at least one $f \in W$ which carries a μ^*-measurable set to a non-μ^*-measurable set (assuming, of course, that there exists at least one non-μ^*-measurable set).

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