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IMMERSIONS AND EMBEDDINGS OF PROJECTIVE SPACES

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ABSTRACT. The topic is embedding and immersions of complex and quaternionic projective spaces. The results are obtained using spin representations and relating these with the various K-theories in which they occur. A numerical result on nonembeddings and and nonimmersions is obtained.

The purpose of this note is to revise and complete the results of **[1]** and **[2]**. The reader is referred to the aforementioned articles for a statement of the problem, notation and techniques. We recall only that $\not\subseteq$ and $\not\subset$ stand for "does not immerse" and "does not embed" respectively; CP_n denotes the complex and HP_n the quaternion projective space and KG denotes the theory of stable G-bundles.

The complete statement of results would be as follows:

THEOREM 1. *$CP_n \not\subseteq R^{4n-2\alpha(n)-1}$ for all n.*

THEOREM 2. *$CP_n \not\subset R^{4n-2\alpha(n)}$ for all n.*

THEOREM 3. *$HP_n \not\subset R^{8n-2\alpha(n)-2}$; $HP_n \not\subset R^{8n-2\alpha(n)}$ if $\alpha(n) \equiv 0$ (4)[1]; $HP_n \not\subseteq R^{8n-2\alpha(n)-3}$; $HP_n \not\subseteq R^{8n-2\alpha(n)-2}$ if $\alpha(n) \not\equiv 1$ (4) and $HP_n \not\subseteq R^{8n-2\alpha(n)-1}$ if $\alpha(n) \equiv 0$ or 3 (4).*

$\alpha(n)$ is the number of 1's in the dyadic expansion of n.

All the results are obtained using K-theory, namely the spin representation. The improvements come from the knowledge that in various dimensions the complex spin representations are restrictions of quaternionic ones or come from real representations.

Theorem 1 has been proven for n odd. To remove this restriction we consider $\mu = H + \nu$ where ν is the normal bundle of $CP_n \subseteq R^{2n+2k+1}$ (the result is better than that obtained considering $CP_{n-1} \subset CP_n$). Now μ is a spin bundle. We let $x = H + \bar{H} - 2$, x is the generator of the subalgebra of

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[1] We thank the referee for pointing out that this result of K. Mayer also follows from our techniques.

selfconjugate elements in $KU(CP_n)$ and $x^{n/2+1}=0$. We have $\mu=2n+2k+3-nH$ and

$$\lambda_t(\mu_u) = (1 + t)^{2n+2k+3}(1 + tH)^{-n}(1 + t\bar{H})^{-n}.$$

Using the fact that $H\bar{H}=1$ we find $\lambda_t(\mu_u)=(1+t)^{2k+3}(1+tx/(1+t)^2)^{-n}$ and $\lambda_1(\mu_n)=2^{2k+3}(1+x/4)^{-n}$. Therefore $\Delta(\mu)=2^{k+1}(1+x/4)^{-n/2}$ is an element of $KU(CP_n)$. Checking the coefficient of $x^{n/2}$ we get $\pm 2^{k+1}\binom{n-1}{n/2}2^{-n}$. This must be an integer and since the highest power of 2 dividing $\binom{a+b}{a}$ is $\alpha(a)+\alpha(b)-\alpha(a+b)$ we must have

$$k + 1 - n + \alpha(n/2) + \alpha(n/2 - 1) - \alpha(n - 1) \geqq 0.$$

For n even $\alpha(n/2)=\alpha(n)$ and $\alpha(n/2-1)=\alpha(n-1)$ so the last inequality is $k \geqq n-\alpha(n)$.

A similar argument removes the restriction $n\equiv 1$ (2) in Theorem 2; note, however, that the proof as given in [**2**] proves the nonembedding result and not nonimmersion as was claimed. This is because we assumed that $\Delta^+-\Delta^-=0$ which is the case for embeddings but not for immersions in general.

The statements in Theorem 3 which give no restriction on $\alpha(n)$ were obtained in [**2**]. To prove the other part of the statement we need the following results from the theory of group representations (e.g. [**3**, p. 193]):

$$\Delta(\nu) \in \operatorname{Im}(KO(HP_n) \to KU(HP_n))$$

whenever dim $\nu\equiv 0$, 1, 2, 6 or 7 mod 8;

$$\Delta^+(\nu) \in \operatorname{Im}(KO(HP_n) \to KU(HP_n))$$

whenever dim $\nu\equiv 0$ mod 8;

$$\Delta(\nu) \in \operatorname{Im}(KSp(HP_n) \to KU(HP_n))$$

whenever dim $\nu\equiv 2$, 3, 4, 5 or 6 mod 8;

$$\Delta^+(\nu) \in \operatorname{Im}(KSp(HP_n) \to KU(HP_n))$$

whenever dim $\nu\equiv 4$ mod 8.

To prove that $HP_n \nsubseteq R^{8n-2\alpha(n)-2}$ for $\alpha(n)\not\equiv 1$ (4) let us assume that such an immersion were possible; the normal bundle ν would be a $4n-2\alpha(n)-2$ dimensional spin-bundle and we would have $\Delta(\nu) \in KU(HP_n)$ given by

$$\Delta(\nu) = 2^{2n-\alpha(n)-1}(1 + z/4)^{-(n+1)}(1 + z/2).$$

Since the image $KO(HP_n)\to KU(HP_n)$ is generated by $2z$ and z^2 ($z^{n+1}=0$) the result follows for n even. Indeed, $4n-2\alpha(n)-2\equiv 0$, 2 or 6 mod 8 provided $\alpha(n)\not\equiv 1$ (4) and the coefficient of z^{n-1} in $\Delta(\nu)$ is (e.g. [**2**]) $\pm 2^{2n-\alpha(n)-1-2(n-1)}\binom{2n-1}{n}/2n-1$, and this is always an odd number.

For n odd we observe that $4n-2\alpha(n)-2\equiv 2$, 4 or 6 mod 8 if $\alpha(n)\not\equiv 1$ (4) and so

$$\Delta(\nu)\in \operatorname{Im}(KSp(HP_n)\to KU(HP_n)).$$

This image is only a submodule and is generated by z^{2i+1} and $2z^{2i}$. Now $n-1$ is even and the same calculation yields the result.

If $\alpha(n)\equiv 0$ or 3 (4) the above argument gives a contradiction to $HP_n\subseteq R^{8n-2\alpha(n)-1}$.

The nonembedding result $HP_n\not\subset R^{8n-2\alpha(n)}$ provided $\alpha(n)\equiv 0$ (4) is obtained by the above argument for $\Delta^+(\nu)=\frac{1}{2}\Delta(\nu)$.

References

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