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NORM REDUCTION OF AVERAGING OPERATORS

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ABSTRACT. Suppose $\phi:S\to T$ is an irreducible map of compact Hausdorff spaces, and $\mu:T\to M(S)$ the integral representation of an averaging operator for ϕ. We obtain an inequality of the form $\|\mu(t)\|\leqq\|\mu\|-a(t)$, where $a(t)$ is a positive number depending on t. From this, some results of Amir and Isbell-Semadeni on P_λ spaces are shown to follow quickly and a theorem on the isomorphism of certain continuous function spaces is derived.

1. **Introduction.** For convenience, we consider only Banach spaces over the real numbers. In all that follows, S and T are compact Hausdorff spaces and $\phi:S\to T$ a continuous onto function. The map ϕ induces a linear isometry, $\phi^\circ(f)\equiv f\circ\phi$, from the sup-norm continuous function space $C(T)$ into $C(S)$, and a left inverse $u:C(S)\to C(T)$ for ϕ° is called an *averaging operator* for ϕ. Any continuous linear operator $u:C(S)\to C(T)$ determines and is determined by a continuous function $t\to\mu(t)$ carrying T into the set $M(S)$ of all regular (signed) Borel measures on S with the weak* topology, and u is an averaging operator for ϕ if and only if $\mu(t)(\phi^{-1}B)=\delta_t(B)$ for each Borel subset B of T. Here δ_t denotes the unit point mass measure at t. The map μ is usually called the integral representation of u since $u(f)(t)=\int f\,d\mu(t)$, but we will refer to it as a *dual map*. In case $\mu(t)(\phi^{-1}B)=\delta_t(B)$ is satisfied for all the Borel subsets of some fixed Borel set V of T, we shall say that the dual map μ *averages* ϕ on V, and if μ averages ϕ on T, we then say that μ averages ϕ.

The map ϕ is called *irreducible* if the following equivalent conditions are fulfilled: (a) if K is a proper closed subset of S, then ϕK is not all of T; (b) if U is a nonempty open subset of S, then U contains $\phi^{-1}V$ for some nonempty open subset V of T; and (c) if U is a nonempty open subset of S then U contains $\phi^{-1}(t)$ for some t in T.

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We will need the following

LEMMA 1. *If μ_α converges to μ in the weak* topology of $M(S)$, then for each open subset U of S,*

$$\liminf_\alpha |\mu_\alpha|(U) \geqq |\mu|(U); \textit{ in particular, } \liminf_\alpha \|\mu_\alpha\| \geqq \|\mu\|.$$

PROOF. Let $\varepsilon>0$ be given. There is a continuous function f that vanishes outside of U, takes all its values in $[-1, 1]$ and $\int f\,d\mu>|\mu|(U)-\varepsilon$. For each α,

$$|\mu_\alpha|(U) = \int \chi_U \, d|\mu_\alpha| \geqq \int |f| \, d|\mu_\alpha| \geqq \int f \, d\mu_\alpha$$

so

$$\liminf_\alpha |\mu_\alpha|(U) \geqq \lim_\alpha \int f\,d\mu_\alpha = \int f\,d\mu > |\mu|(U) - \varepsilon$$

as desired.

The following function defined on the reals occurs in our theorem below. Let $\theta(\lambda)=|1-\lambda|-|\lambda|$. The values of θ are in $[-1, 1]$ with -1 assumed for all $\lambda\geqq 1$, 1 assumed for $\lambda\leqq 0$, and $\theta(\lambda)=1-2\lambda$ for $0\leqq\lambda\leqq 1$.

2. Norm reduction of averaging operators.

LEMMA 2. *Suppose ϕ is irreducible, $\mu: T\to M(S)$ a dual map that averages ϕ on an open set V of T, and $\phi(s)=t \in V$. Then*

$$\|\mu(t)\| \leqq \sup(v \in V)\ \|\mu(v)\| - 1 - \theta(\mu(t)\{s\}).$$

PROOF. First note that for each v in V, $\mu(v)(\phi^{-1}(v))=\delta_v(\{v\})=1$. Let $\varepsilon>0$ be given. Choose open sets W and U such that $s \in W\subseteq \operatorname{cl} W\subseteq U\subseteq \operatorname{cl} U\subseteq \phi^{-1}(V)$ and $|\mu(t)|(\operatorname{cl} U\setminus\{s\})<\varepsilon$. Let $f: S\to[0, 1]$ be continuous with $f\operatorname{cl} W=\{1\}$ and $f[S\setminus U]=\{0\}$. Since ϕ is irreducible, there is a net v_α in V converging to t with $\phi^{-1}(v_\alpha)\subseteq W$ for all α. We have

$$\begin{aligned}
\mu(t)\{s\} + \int_{U\setminus\{s\}} f\,d\mu(t) &= \int f\,d\mu(t) \\
&= \lim_\alpha \int f\,d\mu(v_\alpha) = \lim_\alpha \int_U f\,d\mu(v_\alpha) \\
&= \lim_\alpha \left[\int_{U\setminus\phi^{-1}(v_\alpha)} f\,d\mu(v_\alpha) + \int_{\phi^{-1}(v_\alpha)} f\,d\mu(v_\alpha) \right] = \lim_\alpha \int_{U\setminus\phi^{-1}(v_\alpha)} f\,d\mu(v_\alpha) + 1.
\end{aligned}$$

Consequently $\lim_\alpha \int_{U\setminus\phi^{-1}(v_\alpha)} f\,d\mu(v_\alpha)$ exists and is equal to $\mu(t)\{s\}-1+\int_{U\setminus\{s\}} f\,d\mu(t)$. Also $\liminf_\alpha |\mu(v_\alpha)|(S\setminus\operatorname{cl} U)\geqq|\mu(t)|(S\setminus\operatorname{cl} U)$ by the Lemma 1 above.

Therefore,

$$\begin{aligned}
\sup(v \in V)\, \|\mu(v)\| &\geqq \liminf_{\alpha} \|\mu(v_\alpha)\| \\
&= \liminf_{\alpha} [|\mu(v_\alpha)|\,(\phi^{-1}(v_\alpha)) + |\mu(v_\alpha)|\,(\mathrm{cl}\, U \backslash \phi^{-1}(v_\alpha)) + |\mu(v_\alpha)|\,(S \backslash \mathrm{cl}\, U)] \\
&\geqq 1 + \liminf_{\alpha} |\mu(v_\alpha)|\,(U \backslash \phi^{-1}(v_\alpha)) + |\mu(t)|\,(S \backslash \mathrm{cl}\, U) \\
&\geqq 1 + \lim_{\alpha} \left| \int_{U \backslash \phi^{-1}(v_\alpha)} f\, d\mu(v_\alpha) \right| + |\mu(t)|\,(S \backslash \mathrm{cl}\, U) \\
&= 1 + \left| (1 - \mu(t)\{s\}) - \int f\, d\mu(t) \right| \\
&\quad + \|\mu(t)\| - |\mu(t)|\,(\mathrm{cl}\, U \backslash \{s\}) - |\mu(t)\{s\}| \\
&\geqq 1 + \|\mu(t)\| + \theta(\mu(t)\{s\}) - \left| \int_{U \backslash \{s\}} f\, d\mu(t) \right| - |\mu(t)|\,(\mathrm{cl}\, U \backslash \{s\}) \\
&\geqq 1 + \|\mu(t)\| + \theta(\mu(t)\{s\}) - 2\varepsilon
\end{aligned}$$

which completes the proof.

No new information is obtained from Lemma 2 when $\phi^{-1}(t)$ is a singleton $\{s\}$ whereupon $\theta(\mu(t)\{s\}) = -1$. If $\phi^{-1}(t)$ is infinite, then $\mu(t)\{s\}$ can be brought close to 0 and so $\|\mu\| \geqq \sup(v \in V)\|\mu(v)\| > \|\mu(t)\| + 2 \geqq 3$. If $\phi^{-1}(t)$ has a finite number $n \geqq 2$ of points, then $\mu(t)\{s\} \leqq 1/n$ for some s in $\phi^{-1}(t)$, whence $\theta(\mu(t)\{s\}) \geqq \theta(1/n) = 1 - 2/n$ and so

$$\|\mu\| \geqq \sup(v \in V)\|\mu(v)\| \geqq 3 - 2/n.$$

COROLLARY 1. *If $\phi: S \to T$ is an onto irreducible map admitting an averaging operator, then the set $K_2 \equiv \{t \in T : \mathrm{card}\, \phi^{-1}(t) \geqq 2\}$ of plural points of ϕ is nowhere dense in T.*

PROOF. Let $\mu: T \to M(S)$ be a dual map that averages ϕ. What if cl K_2 contains a nonvoid open subset V?! If t is a plural point in V, then $\theta(\mu(t)\{s\}) \geqq 0$ for some $s \in \phi^{-1}(t)$ so that $\|\mu(t)\| \leqq \sup(v \in V)\|\mu(v)\| - 1$ by Lemma 2. If v' is any element of V, then $v' = \lim_\alpha t_\alpha$ for some net t_α of plural points in V. Consequently

$$\|\mu(v')\| \leqq \liminf_{\alpha} \|\mu(t_\alpha)\| \leqq \sup(v \in V)\|\mu(v)\| - 1.$$

Thus $\sup(v' \in V)\|\mu(v')\| \leqq \sup(v \in V)\|\mu(v)\| - 1$, a contradiction.

COROLLARY 2 (AMIR, [**1**]). *If $C(T)$ is a P_λ space, then T has an open dense extremally disconnected subset.*

PROOF. Let $\phi: S \to T$ be Gleason's map. By Corollary 1, $V \equiv T \backslash \mathrm{cl}\, K_2$ is a dense open subset of T and $U \equiv \phi^{-1}V$ is an extremally disconnected topological space which ϕ carries 1-1 onto V. We need only show that on U, ϕ is an open mapping. Let W be an open subset of U. If ϕW is not open, there is a $w \in W$ and a net $t_\alpha \in T \backslash \phi W$ such that $t_\alpha \to \phi(w)$. Let $s_\alpha \in S$ such that $\phi(s_\alpha) = t_\alpha$ for all ϕ. Then $s_\alpha \in S \backslash W$ which is compact so that, passing to a convergent subnet without changing notation, we can assume s_α converges to some s not in W. By continuity of ϕ, $\phi(s) = \lim t_\alpha = \phi(w)$ which is not in K_2. But $s \neq w$, a contradiction.

COROLLARY 3 (AMIR [**1**], ISBELL AND SEMADENI [**3**]). *If $C(T)$ is a P_λ space for $\lambda < 3$, and n is the largest natural number such that $3 - 2/n \leqq \lambda$, then* card *$\phi^{-1}(t) \leqq n$ for every t in T, where $\phi: S \to T$ is Gleason's map. So if $\lambda < 2$, T is extremally disconnected.*

PROPOSITION. *Let S_α be a nontrivial compact metric space for each α in an infinite index set A. Let $S \equiv \Pi(\alpha \in A)S_\alpha$ and T a compact Hausdorff space. If there is an irreducible onto map $\phi: S \to T$ admitting an averaging operator, then $C(S)$ and $C(T)$ are isomorphic.*

PROOF. Let S_1 denote the generalized Cantor space $\{0,1\}^A$. By [**4**, Theorem 8.8], $C(S_1)$ is isomorphic to $C(S)$. We will show that $C(S_1)$ is isomorphic to $C(T)$. Since $C(T)$ is isometric to $\phi^\circ C(T)$, a complemented subspace of $C(S)$, it follows that $C(T)$ is isomorphic to a complemented subspace of $C(S_1)$. Hence by [**4**, Proposition 8.3], it suffices to show that $C(S_1)$ is isomorphic to a complemented subspace of $C(T)$.

By Corollary 1 above there is a nonempty open set U in S such that ϕ is one-one on U. U contains a set of the form $\Pi(\alpha \in A)\{s_1^\alpha, s_2^\alpha\}$ where $s_1^\alpha, s_2^\alpha \in S_2$ for each $\alpha \in A$ and $s_1^\alpha \neq s_2^\alpha$ for all but finitely many $\alpha \in A$. There is a norm preserving extension operator from $C(\phi(\Pi(\alpha \in A)\{s_1^\alpha, s_2^\alpha\}))$ into $C(T)$, [**4**, Theorem 6.6], since ϕ is 1-1 on U. Hence, $C(\phi(\Pi(\alpha \in A)\{s_1, s_2\}))$ is isometric to a complemented subspace of $C(T)$. Since S is homeomorphic to $\phi(\Pi(\alpha \in A)\{s_1^\alpha, s_2^\alpha\})$, it follows that $C(S_1)$ is isometric to a complemented subspace of $C(T)$, as desired.

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