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NODAL ALGEBRAS DEFINED BY SKEW-SYMMETRIC BILINEAR FORMS

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ABSTRACT. This paper studies nodal algebras defined by skew-symmetric bilinear forms, a subclass of the class of Kokoris algebras. The ideals of such algebras are classified and a characterization of the automorphisms of these algebras is given.

1. **Introduction.** The purposes of this paper are to specify the ideal structure and to furnish a characterization of the automorphisms of nodal algebras defined by skew-symmetric bilinear forms. Each of these algebras is a member of the following class, $\mathscr{K}$, of Kokoris algebras.

Let the commutative associative truncated polynomial algebra $B_{n,p}(F)$ be defined by

$$B_{n,p}(F) = \frac{F[1, X_1, \cdots, X_n]}{(X_1^p, \cdots, X_n^p)},$$

for F a field of characteristic $p>2$. An algebra A is in $\mathscr{K}$ if and only if there exist p, n, and F such that $A^+=B_{n,p}(F)$ where the product of $f, g \in A$ is defined in terms of the (dot) product of A^+ as

$$fg = f \cdot g + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot [x_i, x_j], \tag{1}$$

where at least one commutator $[x_i, x_j]=x_ix_j-x_jx_i$ is nonsingular, and where x_i is the coset $X_i+(X_1^p, \cdots, X_n^p)$. Then necessarily, $n\geqq 2$ and A is a nodal noncommutative Jordan algebra [7].

Let A be an algebra in $\mathscr{K}$ of dimension p^n and let N be the radical of $B_{n,p}(F)$. As in R. D. Schafer [**10**], we say that A *is defined by the skew-symmetric bilinear form* ϕ if and only if there are generators $x_1, \cdots, x_n$ of A such that ϕ acting on the n-dimensional F-vector space $N/N\cdot N\approx Fx_1+\cdots+Fx_n$ satisfies $\phi(x_i, x_j)1=\frac{1}{2}[x_i, x_j]\in F1$ for $1\leqq i,j\leqq n$. In [**10**], Schafer represented certain simple Lie algebras of characteristic p as ideals in the derivation algebras of suitable algebras defined by nondegenerate

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skew-symmetric bilinear forms. In [**8**], R. H. Oehmke makes essential use of Schafer's determination of the derivation algebra of an algebra defined by a skew-symmetric bilinear form to find the derivation algebras of certain simple, Lie-admissible members of $\mathcal{K}$.

Suppose A with n generators is defined by a skew-symmetric bilinear form ϕ of rank $2r$ ($2 \leqq 2r \leqq n$). The canonical form of the matrix of ϕ is well known [**5**, pp. 159ff.], and a change of basis in the vector space $Fx_1 + \cdots + Fx_n$ gives

$$\begin{aligned} \phi(x_i, x_{i+r}) &= 1 = -\phi(x_{i+r}, x_i), & i &= 1, \cdots, r; \\ \phi(x_i, x_j) &= 0, & &\text{otherwise.} \end{aligned} \tag{2}$$

A is simple if and only if $n = 2r$ ([**7**], [**10**]). We will extend the methods of [**3**] and [**4**] to prove the following theorem.

THEOREM 1. *Let A be a p^n-dimensional algebra defined by a skew-symmetric bilinear form of rank $2r$. There exists a set of generators $x_1, \cdots, x_n$ for A such that any nonzero ideal of A is either one of the ideals $x_{2r+1}^{e_{2r+1}} \cdot \ldots \cdot x_n^{e_n} \cdot A$ ($0 \leqq e_j \leqq p-1$ for $2r+1 \leqq j \leqq n$) or a sum of such ideals.*

2. **Preliminary lemmas.** We need the following lemma due to A. A. Albert.

LEMMA 1 [**1**, p. 340]. *Let $A = F1 + F[x_1, \cdots, x_n]$ be in $\mathcal{K}$ and let I be a nonzero ideal of A. Then the maximal degree monomial $x_1^{p-1} \cdot x_2^{p-1} \cdot \ldots \cdot x_n^{p-1} \in I$.*

LEMMA 2. *Let A be a p^n-dimensional algebra defined by a skew-symmetric bilinear form ϕ of rank $2r$. Let $x_1, \cdots, x_n$ be generators for A such that ϕ satisfies (2). Then each of the subspaces $x_{2r+1}^{e_{2r+1}} \cdot \ldots \cdot x_n^{e_n} \cdot A$ for $0 \leqq e_k \leqq p-1$ ($2r+1 \leqq k \leqq n$) is an ideal of A.*

PROOF. Each of these subspaces is an ideal of A^+. It suffices to show that each subspace is also an ideal of A^-. Let a and g be arbitrary elements of A. Since $fg = f \cdot g + \frac{1}{2}[f, g]$, (1) and the fact that A is defined by ϕ imply that $[f, g] = 2 \sum_{i,j=1}^{n} (\partial f/\partial x_i) \cdot (\partial g/\partial x_j) \cdot \phi(x_i, x_j)1$, for any $f \in A$.

Recall that for each $g \in A$ the map $D(g): f \mapsto [f, g]$ is a derivation on A^+. Thus, if we set $f = x_{2r+1}^{e_{2r+1}} \cdot \ldots \cdot x_n^{e_n}$,

$$\begin{aligned} [f \cdot a, g] &= (f \cdot a)D(g) = f \cdot [a, g] + a \cdot [f, g] \\ &\equiv a \cdot [f, g] \pmod{f \cdot A} \\ &\equiv 2a \cdot \left[\sum_{i=1}^{r} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_{i+r}} - \sum_{i=r+1}^{2r} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_{i-r}} \right] \pmod{f \cdot A} \text{ by (2)} \\ &\equiv 0 \pmod{f \cdot A} \text{ because } f \text{ is independent of } x_1, \cdots, x_{2r}. \end{aligned}$$

Therefore, $f \cdot A$ is an ideal of A^- and Lemma 2 is proved.

There are p^{n-2r} ideals listed in Lemma 2, including A itself. Sums of these ideals are, of course, also ideals of A. The remainder of our work involves showing every ideal of A can be so described.

3. **The case** $n-2r=1$. In this section $A=F1+F[x_1, \cdots, x_{2r+1}]$ will be an algebra defined by a skew-symmetric bilinear form ϕ of rank $2r$ with generators chosen so as to satisfy (2). Our object is to prove that a complete list of the nonzero ideals of A is given by $A, x_{2r+1} \cdot A, x_{2r+1}^2 \cdot A, \cdots, x_{2r+1}^{p-1} \cdot A$. The method of proof is basically that of [**4**], so we will be brief. Suppose I is a nonzero ideal of A. Lemma 1 implies that $m=x_1^{p-1} \cdot \ldots \cdot x_{2r+1}^{p-1} \in I$. I is an ideal of A^-, consequently,

$$2^{r(1-p)}[(p-1)!]^{-r} m D^{p-1}(x_{1+r}) \cdots D^{p-1}(x_{2r}) = x_{r+1}^{p-1} \cdot \ldots \cdot x_{2r+1}^{p-1}$$

is in I. A similar iterated application of the derivations $D(x_1), \cdots, D(x_r)$ to this last monomial yields $x_{2r+1}^{p-1} \in I$. Thus, $x_{2r+1}^{p-1} \cdot A \subseteq I$; that is, $x_{2r+1}^{p-1} \cdot A$ is the unique minimal ideal of A which is contained in every nonzero ideal of A.

Denote $A/x_{2r+1}^{p-1} \cdot A$ by $A(p-1)$. If $\pi: A \rightarrow A(p-1)$ is the natural homomorphism of A onto $A(p-1)$, then $A(p-1)$ is a nodal algebra [**9**] which inherits the basic multiplicative structure of A and, thus, is an algebra defined by a skew-symmetric bilinear form. The monic maximal degree monomial of $A(p-1)$ is $(x_1\pi)^{p-1} \cdot \ldots \cdot (x_{2r}\pi)^{p-1} \cdot (x_{2r+1}\pi)^{p-2}$ where we again use dots to indicate product in $A(p-1)^+$. Using the same reasoning we applied to A, we see that $(x_{2r+1}\pi)^{p-2} \cdot A(p-1)$ is the unique minimal ideal of $A(p-1)$ contained in every ideal of $A(p-1)$. We now indulge in the notational luxury of dropping the symbol "π" here and refer to $x_{2r+1}^{p-2} \cdot A(p-1)$ as the unique minimal ideal of $A(p-1)$. The possible confusion introduced by this notational convention, which we adopt throughout, is small and is outweighed by its advantages in keeping the notation manageable.

We now recursively define a sequence of algebras defined by a skew-symmetric bilinear form: $A(p-1), \cdots, A(1)$. Having defined $A(p-i)$, with unique minimal ideal $x_{2r+1}^{p-i-1} \cdot A(p-i)$, we proceed to define

$$A(p-i-1) = A(p-i)/x_{2r+1}^{p-i-1} \cdot A(p-i).$$

The unique minimal ideal of $A(p-i-1)$ is $x_{2r+1}^{p-i-2} \cdot A(p-i-1)$.

Since we reduce the maximum exponent of x_{2r+1} by one with each step down this sequence of nodal algebras, it is clear that $A(1) \approx F1+F[x_1, \cdots, x_{2r}]$, an algebra defined by a skew-symmetric bilinear form of rank $2r$. Thus, $A(1)$ is simple, but $A(1)=A(2)/x_{2r+1} \cdot A(2)$, which, given the inclusion property of the denominator ideal, implies that $x_{2r+1} \cdot A(2)$ is the only proper ideal of $A(2)$. Similarly, $x_{2r+1} \cdot A(3)$ and $x_{2r+1}^2 \cdot A(3)$ are the only proper ideals of $A(3)$, and so on up the sequence back to A and the conclusion that $x_{2r+1} \cdot A, \cdots, x_{2r+1}^{p-1} \cdot A$ are the only proper ideals of A. The lattice of ideals of A is linearly ordered, so we form no new ideals by taking sums of these.

4. **The case $n-2r=2$.** In this section $A=F1+F[x_1, \cdots, x_{2r+2}]$ will be an algebra defined by a skew-symmetric bilinear form ϕ of rank $2r$ with generators chosen so as to satisfy (2). It is worthwhile to discuss this case separately for the reason that although we repeat the above process of taking quotients by minimal ideals at various stages, the allowable set of minimal ideals no longer contains just one member at each stage. Discussion of this case will reveal all the features of the general case and provide an insight into it unencumbered by too much notation.

Just as in §3, one can show that $x_{2r+1}^{p-1} \cdot x_{2r+2}^{p-1} \cdot A$ is the unique minimal ideal of A contained in any nonzero ideal of A. Denote $A/x_{2r+1}^{p-1} \cdot x_{2r+2}^{p-1} \cdot A$ by $A(p-1,p-1)$. One has the natural homomorphism of A onto $A(p-1,p-1)$. Thus, $A(p-1,p-1)$ is an algebra defined by a skew-symmetric bilinear form which has two monic monomials of maximal degree: namely, $x_1^{p-1} \cdot \ldots \cdot x_{2r}^{p-1} \cdot x_{2r+1}^{p-2} \cdot x_{2r+2}^{p-1}$ and $x_1^{p-1} \cdot \ldots \cdot x_{2r}^{p-1} \cdot x_{2r+1}^{p-1} \cdot x_{2r+2}^{p-2}$. The method of proof of Lemma 1 can be adapted here, to permit the observation that at least one of these monomials is contained in any nonzero ideal of $A(p-1,p-1)$. Thus, using the method of §3, we find that $x_{2r+1}^{p-2} \cdot x_{2r+2}^{p-1} \cdot A(p-1,p-1)$ and $x_{2r+1}^{p-1} \cdot x_{2r+2}^{p-2} \cdot A(p-1,p-1)$ are minimal ideals of $A(p-1,p-1)$ at least one of which is contained in any nonzero ideal of $A(p-1,p-1)$.

If $A(p-2,p-1)$ denotes $A(p-1,p-1)/x_{2r+1}^{p-2} \cdot x_{2r+2}^{p-1} \cdot A(p-1,p-1)$ and $A(p-1,p-2)$ denotes $A(p-1,p-1)/x_{2r+1}^{p-1} \cdot x_{2r+2}^{p-2} \cdot A(p-1,p-1)$, then there are two maps out of $A(p-1,p-1)$ to consider: the natural homomorphism of $A(p-1,p-1)$ onto $A(p-2,p-1)$ and the natural homomorphism of $A(p-1,p-1)$ onto $A(p-1,p-2)$. If one were to continue this process of forming new algebras defined by bilinear forms, taking each of the possible quotients by minimal ideals and reducing the maximal exponent of either x_{2r+1} or x_{2r+2} by one at each level, then one would obtain a tree of homomorphisms. More specifically, assuming that the algebra $A(p-i,p-j)$ has been defined and has at least one of the minimal ideals $x_{2r+1}^{p-i-1} \cdot x_{2r+2}^{p-j} \cdot A(p-i,p-j)$ or $x_{2r+1}^{p-i} \cdot x_{2r+2}^{p-j-1} \cdot A(p-i,p-j)$ contained in each nonzero ideal, we define

$$A(p-i-1, p-j) = \frac{A(p-i, p-j)}{x_{2r+1}^{p-i-1} \cdot x_{2r+2}^{p-j} \cdot A(p-i, p-j)}$$

and

$$A(p-i, p-j-1) = \frac{A(p-i, p-j)}{x_{2r+1}^{p-i} \cdot x_{2r+2}^{p-j-1} \cdot A(p-i, p-j)}.$$

The natural epimorphisms of $A(p-i,p-j)\to A(p-i-1,p-j)$ and $A(p-i,p-j)\to A(p-i,p-j-1)$ are branches leading to two new nodes on the next level of the tree. As one continues to follow branches down to new levels, the components α, β of $A(\alpha,\beta)$ decrease and the nodes at the end of each branch will for the first time eventually assume either the form $A(0,\beta)$ or $A(\alpha,0)$. Nodes of either of these forms will terminate our construction of the tree from their particular branches, since the case $n-2r=1$ above guarantees that the only nonzero ideals of $A(0,\beta)$ are $x_{2r+2}^{e}\cdot A(0,\beta)$ for $0\leqq e\leqq\beta-1$ and that the only nonzero ideals of $A(\alpha,0)$ are $x_{2r+1}^{e}\cdot A(\alpha,0)$ for $0\leqq e\leqq\alpha-1$. Of course, a node of the form $A(0,0)$ is simple ($n=2r$ here). Once one knows a complete list of ideals of some terminal node of the tree, one can pull this list back (take inverse images) and make use of the inclusion properties of the minimal denominator ideals to determine a complete list of the ideals of the parent algebras.

One continues back up the tree to emerge with a complete list of ideals of A. It is clear from the construction of the homomorphism tree that the nonzero ideals of A are precisely

$$x_{2r+1}^{e_{2r+1}}\cdot x_{2r+2}^{e_{2r+2}}\cdot A,\qquad 0\leqq e_{2r+1},e_{2r+2}\leqq p-1,$$

and sums of these ideals. For the construction process reduced exponents of either x_{2r+1} or x_{2r+2} by one for each level down until a terminal level algebra whose ideals were known was reached. The taking of inverse images and addition of the denominator ideals just increases such exponents by one, enlarging the list of ideals at each step.

5. **The general case.** Let $A=F1+F[x_1,\cdots,x_n]$ be an algebra defined by a skew-symmetric bilinear form ϕ of rank $2r$ with generators chosen so as to satisfy (2). Theorem 1 is proved by induction on $n-2r$. From the preceding, the truth of the theorem is clear for $n-2r=0$, 1, and 2. As induction hypothesis, assume the truth of the theorem for all algebras defined by skew-symmetric bilinear forms with number of generators minus rank less than $n-2r$.

As in previous sections one can prove that $x_{2r+1}^{p-1}\cdot\ \ldots\ \cdot x_n^{p-1}\cdot A$ is the unique minimal ideal of A contained in every nonzero ideal of A. Set $A(p-1,\cdots,p-1)$ ($n-2r$ components, each $=p-1$) equal to $A/x_{2r+1}^{p-1}\cdot\ldots\cdot x_n^{p-1}\cdot A$.

Just as in §4, we can conclude that all

$$x_{2r+1}^{p-1}\cdot\ldots\cdot x_{k-1}^{p-1}\cdot x_k^{p-2}\cdot x_{k+1}^{p-1}\cdot\ldots\cdot x_n^{p-1}\cdot A(p-1,\cdots,p-1),\qquad 2r+1\leqq k\leqq n,$$

are minimal ideals of $A(p-1,\cdots,p-1)$ at least one of which is contained in every nonzero ideal of $A(p-1,\cdots,p-1)$. We now can construct $n-2r$ natural homomorphisms out of $A(p-1,\cdots,p-1)$, each one having one of the $n-2r$ minimal ideals above as kernel. This starts the construction of a tree of homomorphisms in a manner similar to that of §4.

Again leaning upon the analysis in §4, it is clear that we could define an $(n-2r)$-fold sequence of algebras $A(i_1, \cdots, i_{n-2r})$ $(0 \leqq i_j \leqq p-1)$ defined by skew-symmetric bilinear forms. Eventually, in proceeding down branches of a homomorphism tree having these algebras as nodes, some component i_j in $A(i_1, \cdots, i_{n-2r})$ will be zero. In this case, $A(i_1, \cdots, i_{n-2r})$ is an algebra defined by a bilinear form of rank $2r$ with at most the generators $x_1, \cdots, x_{2r}, \cdots, x_{2r+j-1}, x_{2r+j+1}, \cdots, x_n$. By the induction hypothesis we know the ideals of $A(i_1, \cdots, i_{n-2r})$. We then work our way up the branches of the tree in the manner described previously. We have seen that the method of construction of the tree guarantees a complete list of ideals of A is provided in the theorem. This proves the theorem.

6. **Remark.** Theorem 1 and its method of proof actually apply to a larger class of nodal algebras in $\mathscr{K}$ than those defined by skew-symmetric bilinear forms. A careful analysis of the proof shows that any choice of the products $[x_i, x_j]$ which yield Lemma 2 and the second paragraph of §5 (thus, the initial parts of §§3 and 4) will produce the same theorem. For example, if we define:

$$[x_i, x_{i+r}] = \text{any invertible element of } A^+, \qquad 1 \leqq i \leqq r,$$

$$[x_i, x_{i-r}] = \text{any invertible element of } A^+, \qquad r+1 \leqq i \leqq 2r,$$

$$[x_{2r+j}, x_k] = x_{2r+1} \cdot x_{2r+2} \cdot \ldots \cdot x_n \cdot a_j, \qquad a_j \in A,\ 1 \leqq j \leqq n-2r,$$
$$k \neq 2r+j,\ 2r+1 \leqq k \leqq n,$$

and all other $[x_i, x_j]=0$, then every other step in this proof remains valid.

7. **Automorphisms.** Suppose $A \in \mathscr{K}$. It is clear from the relation $fg = f \cdot g + \frac{1}{2}[f, g]$ that $W \in \mathrm{Aut}(A)$, the group of automorphisms of A if and only if $W \in \mathrm{Aut}(A^+) \cap \mathrm{Aut}(A^-)$. Moreover, the automorphisms of A^+ are known modulo $N \cdot N$ from the following lemma due to N. Jacobson.

Lemma 3 [6, pp. 116–117]. *If $A \in \mathscr{K}$ with $A^+ = B_{n,p}(F)$, then*

$$\mathrm{Aut}(A^+)/T \approx \mathrm{GL}(n, F),$$

where $T = \{W \in \mathrm{Aut}(A^+) \mid x_i W \equiv x_i \pmod{N \cdot N}\}$ *and* $\mathrm{GL}(n, F)$ *is the general linear group of degree n over F.*

We then see from Lemma 3 that $W \in \operatorname{Hom}(A^+, A^+)$ is in $\operatorname{Aut}(A^+)$ if and only if

$$(3) \qquad x_i W = \sum_{j=1}^{n} \omega_{ij} x_j + w_i, \qquad w_i \equiv O(N \cdot N), \qquad i = 1, \cdots, n,$$

for $A^+ = B_{n,p}(F)$ and $(\omega_{ij}) \in \mathrm{GL}(n, F)$.

The multiplication defined in (1) yields the fact that $W \in \operatorname{Aut}(A^+)$ is also in $\operatorname{Aut}(A)$ if and only if $[x_i, x_j]W = [x_i W, x_j W]$ for all i, j since it follows readily that $\partial(fW)/\partial(x_i W) = (\partial f/\partial x_i)W$.

Now let A be an algebra defined by a skew-symmetric bilinear form ϕ of rank $2r$, with generators $x_1, \cdots, x_n$ satisfying (2). If $W \in \operatorname{Aut}(A)$, then setting $\phi_{ij} = \phi(x_i, x_j)$ we see from (3) that

$$(4) \qquad \begin{aligned} \phi_{ij} 1 &= \tfrac{1}{2}[x_i, x_j]W = \tfrac{1}{2}[x_i W, x_j W] \\ &= \tfrac{1}{2} \sum_{k,m} \omega_{ik}\omega_{jm}[x_k, x_m] + \tfrac{1}{2} \sum_{k} \omega_{ik}[x_k, w_j] \\ &\quad + \tfrac{1}{2} \sum_{m} \omega_{jm}[w_i, x_m] + \tfrac{1}{2}[w_i, w_j]. \end{aligned}$$

To rewrite (4) we can make use of the observation:

$$\begin{aligned} \tfrac{1}{2}[w_i, w_j] &= \sum_{s,t=1}^{n} \frac{\partial w_i}{\partial x_s} \cdot \frac{\partial w_j}{\partial x_t} \cdot \phi_{s,t} 1 \\ &= \sum_{s=1}^{r} \frac{\partial w_i}{\partial x_s} \cdot \frac{\partial w_j}{\partial x_{s+r}} - \sum_{s=r+1}^{2r} \frac{\partial w_i}{\partial x_s} \cdot \frac{\partial w_j}{\partial x_{s-r}} \\ &= \sum_{s=1}^{r} \left(\frac{\partial w_i}{\partial x_s} \cdot \frac{\partial w_j}{\partial x_{s+r}} - \frac{\partial w_i}{\partial x_{s+r}} \cdot \frac{\partial w_j}{\partial x_s} \right) \\ &= \sum_{s=1}^{r} \left[\frac{\partial}{\partial x_s}\left(w_i \cdot \frac{\partial w_j}{\partial x_{s+r}} \right) - \frac{\partial}{\partial x_{s+r}}\left(w_i \cdot \frac{\partial w_j}{\partial x_s} \right) \right]. \end{aligned}$$

Note that each component of the above sum is antisymmetric in i and j. Similar calculations imply that

$$\begin{aligned} \frac{1}{2} \sum_{k} \omega_{ik}[x_k, w_j] &= \sum_{k=1}^{r} \omega_{ik} \frac{\partial w_j}{\partial x_{k+r}} - \sum_{k=r+1}^{2r} \omega_{ik} \frac{\partial w_j}{\partial x_{k-r}} \\ &= \sum_{k=1}^{r} \left(\omega_{ik} \frac{\partial w_j}{\partial x_{k-r}} - \omega_{i,k+r} \frac{\partial w_j}{\partial x_k} \right). \end{aligned}$$

Thus, (4) becomes

$$\phi_{ij}1 = \sum_{k,m=1}^{n} \omega_{ik}\phi_{km}\omega_{jm}1 \tag{5}$$
$$+ \sum_{k=1}^{r} \left[\frac{\partial}{\partial x_k}\left(\omega_{j,k+r}w_i - \omega_{i,k+r}w_j + w_i \cdot \frac{\partial w_j}{\partial x_{k+r}}\right) + \frac{\partial}{\partial x_{k+r}}\left(\omega_{ik}w_j - \omega_{jk}w_i - w_i \cdot \frac{\partial w_j}{\partial x_k}\right)\right].$$

We can equate components in $F1$ and N separately in (5) to obtain the proof of the next result.

THEOREM 2. *Let A be a p^n-dimensional algebra defined by a skew-symmetric bilinear form ϕ of rank $2r$ with generators $x_1, \cdots, x_n$ chosen so as to satisfy* (2). *If $W \in \mathrm{Hom}(A, A)$ with the action of W upon the generators given by* (3), *then $W \in \mathrm{Aut}(A)$ if and only if*

$$\Phi = \Omega\Phi\Omega^t \quad \textit{and} \quad \sum_{k=1}^{2r} \frac{\partial v_k(i,j)}{\partial x_k} = 0 \quad \textit{for all } 1 \leqq i < j \leqq n,$$

where

$$\Phi = (\phi_{ij}), \qquad \Omega = (\omega_{ij}) \in \mathrm{GL}(n, F),$$

and

$$v_k(i,j) = \omega_{j,k+r}w_i - \omega_{i,k+r}w_j + w_i \cdot \frac{\partial w_j}{\partial x_{k+r}} \qquad (1 \leqq k \leqq r),$$

$$v_k(i,j) = \omega_{i,k-r}w_j - \omega_{j,k-r}w_i - w_i \cdot \frac{\partial w_j}{\partial x_{k-r}} \qquad (r+1 \leqq k \leqq 2r).$$

For $v=(v_1, \cdots, v_n) \in A \times A \times \cdots \times A$ (n-times), we can define divergence and write $\mathrm{div}(v) = \sum_{i=1}^{n} (\partial v_i/\partial x_i)$ ([**2**], [**10**]). When ϕ is a nondegenerate form, we can restate the conditions in Theorem 2.

COROLLARY. *In the notation of Theorem* 2, *if $n=2r$, then $W \in \mathrm{Hom}(A, A)$ is in $\mathrm{Aut}(A)$ if and only if Ω is in the symplectic group of the form ϕ and $\mathrm{div}(v(i,j))=0$, where $v(i,j)=(v_1(i,j), \cdots, v_n(i,j))$ for all $1 \leqq i < j \leqq n$.*

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