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JOIN-PRINCIPAL ELEMENTS IN NOETHER LATTICES

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ABSTRACT. In this paper we determine the structure of join-principal elements in Noether lattices and we apply these results to obtain the Krull Principal Ideal Theorem for join-principal elements, a representation theorem for a class of Noether lattices, and some interesting ring results.

The concept of a Noether lattice, introduced by R. P. Dilworth [1], is an abstraction of the lattice of ideals L of a Noetherian ring R. Fundamental to this abstraction is the following lattice-theoretic characterization of two key properties of a principal ideal E: $(A\wedge B:E)E=AE\wedge B$ and $(A\vee BE):E=(A:E)\vee B$ for all $A, B\in L$. An element E of a multiplicative lattice $\mathscr{L}$ is said to be a *meet-principal* (*join-principal*) element if it satisfies the first (second) identity. E is *principal* if it satisfies both identities.

The concepts of meet- and join-principalness have been instrumental in the generalization of classical results of the ideal theory of Noetherian rings to lattices ([1], [2], and [4]), and have provided a basis for representation and embedding theorems for certain classes of Noether lattices ([3], [4], and [5]). The results of [3], [4], and [5] hint that, in general, the connection between meet- and join-principal elements in a Noether lattice may be quite close. In this paper, we obtain a structural result for join-principal elements (Theorem 1) which not only allows us to explore the relationship between meet- and join-principal elements but it also yields as corollaries the Krull Principal Ideal Theorem for join-principal elements, the representation theorem in [4], the result that join-principal maximal elements have minimal bases consisting of independent elements, and some interesting ring theoretic results.

We adopt the terminology of [1].

Let $\mathscr{L}$ be a noether lattice and let M and E be elements of $\mathscr{L}$ such that E is meet-principal and M is maximal. Then $\{E\}$ is meet-principal, and therefore join-irreducible, in $\mathscr{L}_M$. It follows that $\{E\}$ is principal in $\mathscr{L}_M$,

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and hence that $((B \vee CE):E)_M = ((B:E) \vee C)_M$, for all $B, C \in \mathscr{L}$. Since this is true for every maximal element M, this establishes the following: *If E is a meet-principal element of a Noether lattice $\mathscr{L}$, then E is join-principal.* The converse of this result is, however, not valid in general. To see this, let $(\mathscr{L}_i, M_i)$ be local Noether lattices $(1 \leqq i \leqq n)$, and let $\mathscr{L}$ be the sub-multiplicative-lattice of $\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_n$ defined by $\mathscr{L} = (M_1/O \oplus \cdots \oplus M_n/O) \cup \{(I, \cdots, I)\}$. Then $\mathscr{L}$ is a Noether lattice in which an element $A = (A_1, \cdots, A_n)$ $(A \neq I)$ is join-principal if and only if each of the coordinates A_i is join-principal $(i = 1, \cdots, n)$. On the other hand, A is meet-principal if and only if each of the coordinates A_i is meet-principal and at most one of the A_i is different from 0. Hence, this method of construction defines a large class of Noether lattices which have many join-principal elements that are not meet-principal.

We note that, in the above example, any join-principal element $A = (A_1, \cdots, A_n)$ is decomposable into coordinates $(0, \cdots, A_i, \cdots, 0)$ any two of which annihilate each other. Theorem 1 shows that this is a basic structural property of join-principal elements.

THEOREM 1. *Let $\mathscr{L}$ be a local Noether lattice and let A be a join-principal element of $\mathscr{L}$. Then A has a minimal base $E_1, \cdots, E_k$ such that $E_iE_j = 0$ whenever $i \neq j$.*

PROOF. Let $F_1, \cdots, F_r$ $(r \geqq 0)$ be a minimal collection of principal elements such that $F_i \leqq A \wedge (0:A)$, $i = 1, \cdots, r$, and $\bigvee_{i=1}^r (F_i \vee MA) = (A \wedge (0:A)) \vee MA$. Extend the collection $F_1, \cdots, F_r$ to a minimal base $F_1, \cdots, F_r, \cdots, F_k$ for A, and for each $i = 1, \cdots, k$, set $C_i = F_1 \vee \cdots \vee \hat{F}_i \vee \cdots \vee F_k$. Since A is join-principal and $A^2 = AC_i \vee F_i^2$,

$$A = (A^2:A) \wedge A = ((AC_i \vee F_i^2):A) \wedge A$$
$$= (C_i \vee (F_i^2:A)) \wedge A = C_i \vee ((F_i^2:A) \wedge A).$$

Also,

$$F_i \vee ((F_i^2:A) \wedge A) = ((AF_i \vee F_i^2):A) \wedge A = (AF_i:A) \wedge A$$
$$= (F_i \vee (0:A)) \wedge A = F_i \vee ((0:A) \wedge A).$$

Hence

$$(F_i^2:A) \wedge A = ((F_i^2:A) \wedge A) \wedge (F_i \vee ((0:A) \wedge A))$$
$$= ((0:A) \wedge A) \vee (F_i^2:A) \wedge F_i = ((0:A) \wedge A) \vee (F_i^2:AF_i)F_i,$$

and therefore,

$$A = C_i \vee ((0:A) \wedge A) \vee (F_i^2:AF_i)F_i.$$

Assume $s > r$. Then $(0:A) \wedge A \leqq C_s \vee MA$, so by the Intersection Theorem, $A = C_s \vee (F_s^2:F_sA)F_s$. Since $A \neq C_s$, it follows that $F_s^2:F_sA = I$, and hence that $F_sA=F_s^2$. On the other hand, if $s\leqq r$, then $F_s\leqq 0:A$, and so $F_sA=F_s^2=0$. Therefore $F_iA=F_i^2$, $i=1,\cdots,k$. Hence

$$\begin{aligned} A &= (AF_i:F_i)\wedge A = (F_i^2:F_i)\wedge A \\ &= (F_i\vee(0:F_i))\wedge A = F_i\vee((0:F_i)\wedge A), \qquad i=1,\cdots,k. \end{aligned}$$

Now, assume that s is a nonnegative integer such that $r\leqq s\leqq k-1$ and $(F_1\vee\cdots\vee F_i)(F_{i+1}\vee\cdots\vee F_k)=0$ for all $i=r,\cdots,s$ (if $i=0$, then $F_1\vee\cdots\vee F_i=0$). Then $A=F_{s+1}\vee((0:F_{s+1})\wedge A)$, so $F_{s+1}\vee\cdots\vee F_k=F_{s+1}\vee((F_{s+1}\vee\cdots\vee F_k)\wedge(0:F_{s+1}))$ and $A=F_1\vee\cdots\vee F_{s+1}\vee((F_{s+1}\vee\cdots\vee F_k)\wedge(0:F_{s+1}))$. For $1\leqq i\leqq s+1$, set $F_i'=F_i$. Then $F_1',\cdots,F_{s+1}'$ can be extended to a minimal base $F_1',\cdots,F_k'$ for A using principal elements

$$F_{s+2}',\cdots,F_k'\leqq(F_{s+1}\vee\cdots\vee F_k)\wedge(0:F_{s+1}).$$

Hence $(F_1'\vee\cdots\vee F_i')(F_{i+1}'\vee\cdots\vee F_k')=0$ for each $i=r,\cdots,s+1$. By induction, it now follows that A has a minimal base $E_1,\cdots,E_k$ such that $(E_1\vee\cdots\vee E_i)(E_{i+1}\vee\cdots\vee E_k)=0$ for all $i=0,\cdots,k-1$. Q.E.D.

We now explore some of the consequences of Theorem 1. We begin with an extension of a result of [5] which was instrumental in determining the embeddability of distributive local Noether lattices with join-principal maximal elements.

COROLLARY 1. *Let $(\mathscr{L},M)$ be a local Noether lattice in which M is join-principal. Then M has a minimal base $E_1,\cdots,E_k$ of independent elements.*

PROOF. Let $E_1,\cdots,E_k$ be a minimal base for M such that $E_iE_j=0$ whenever $i\neq j$. We begin by showing that $E_i\wedge E_j=0$ for $i\neq j$. Hence, let E be a principal element such that $E\leqq E_i\wedge E_j$. Assume $E\neq 0$. Choose r such that $E\leqq E_i^r$ and $E\nleqq E_i^{r+1}$. And choose s so that $E\leqq E_j^s$ and $E\nleqq E_j^{s+1}$. Then $E\leqq E_i^r$ and $E\nleqq ME_i^r=E_i^{r+1}$, so $E=E_i^r$. Similarly, $E=E_j^s$. But then $ME_i^{r-1}=ME_j^{s-1}$, so $E_i^{r-1}\vee(0:M)=E_j^{s-1}\vee(0:M)$, and

$$E=E_i^r=E_i(E_i^{r-1}\vee(0:M))=E_i(E_j^{s-1}\vee(0:M))=0.$$

Hence $E_i\wedge E_j=0$. Since M is join-principal in $\mathscr{L}/E_k$, the result follows by induction on k.

COROLLARY 2. *Let $\mathscr{L}$ be a Noether lattice in which, for every maximal element M, O_M is meet-irreducible. If every maximal element is join-principal, then every element of $\mathscr{L}$ is principal and $\mathscr{L}$ is representable as a lattice of ideals of a Noetherian ring.*

PROOF. Since, for every maximal element M, $\{O\}$ is meet-irreducible and $\{M\}$ is join-principal in $\mathscr{L}_M$, it follows that $\{M\}$ is principal in $\mathscr{L}_M$.

Hence, by the results of [3], every element of $\mathscr{L}$ is principal and $\mathscr{L}$ is representable. Q.E.D.

From Corollary 2 and the results of [3], we deduce the following new characterization of general ZPI-rings (i.e., Noetherian multiplication rings): *A Noetherian ring R is a general ZPI-ring if and only if for each maximal ideal M of R, M is join-principal in the lattice of ideals of R and O_M is meet-irreducible.*

In [4] we showed that if a maximal element M is join-principal and 0 is prime, then M is principal. The following Corollary 3 extends this result to an arbitrary join-principal element. Also, Corollary 3 and the results of [3] yield the following new characterization of Dedekind domains: *A Noetherian domain D is Dedekind if and only if every maximal ideal of D is a join-principal element in the lattice of ideals of D.*

COROLLARY 3. *Let $\mathscr{L}$ be a Noether lattice. Let E be an element of $\mathscr{L}$ which is either meet- or join-principal, and let P be a prime element in $\mathscr{L}$. Then $E\vee P$ is principal in $\mathscr{L}/P$. In particular, if 0 is prime in $\mathscr{L}$, then every join-principal element is principal.*

Corollary 3 provides an alternative proof for our extension of the Krull Principal Ideal Theorem to join-principal elements [4]. We state this result as Corollary 4 and we note that since there is a large class of Noether lattices which have many join-principal elements that are not principal, Corollary 4 represents a strengthening of the Principal Ideal Theorem in Noether lattices [1].

COROLLARY 4. *Let $\mathscr{L}$ be a Noether lattice and let E be an element of $\mathscr{L}$ which is either meet- or join-principal. If P is a minimal prime of E, then P has rank at most 1.*

PROOF. The proof is clear, by Corollary 3 and the Principal Ideal Theorem for principal elements [1].

A Noether lattice $\mathscr{L}$ satisfies the *weak union condition* if, given elements A, B, and C such that $A\nleqq B$ and $A\nleqq C$, it follows that there exists a principal element $E\leqq A$ such that $E\nleqq B$ and $E\nleqq C$. This concept was used in [3] to characterize the distributive Noether lattices which are representable. A lattice of ideals of a commutative ring with an identity clearly satisfies this condition.

Let $\mathscr{L}$ be a local Noether lattice. Then $\mathscr{L}$ satisfies the weak union condition if and only if given any principal elements $A_1, \cdots, A_k$ in $\mathscr{L}$, there exists a principal element A such that $A\vee(\bigvee_{i\neq j} A_i)=\bigvee_{i=1}^k A_i$ for all $j=1,\cdots,k$. To see this, assume that $\mathscr{L}$ satisfies the weak union condition, and let A_1 and A_2 be principal elements such that $A_1\nleqq A_2$ and $A_2\nleqq A_1$.

By the Intersection Theorem, $A_1 \nleqq A_2 \vee MA_1$ and $A_2 \nleqq A_1 \vee MA_2$, so there exists a principal element $A \leqq A_1 \vee A_2$ such that $A \nleqq A_2 \vee MA_1$ and $A \nleqq A_1 \vee MA_2$. Then

$$(A_1 \vee A_2)/(A_1 \vee MA_2) \simeq A_2/(A_1 \vee MA_2) \wedge A_2 \\ = A_2/[(A_1 \vee MA_2):A_2]A_2 = A_2/MA_2,$$

and so $(A_1 \vee A_2)/(A_1 \vee MA_2)$ is simple. Hence, $A \vee A_1 \vee MA_2 = A_1 \vee A_2$ and similarly $A \vee A_2 \vee MA_1 = A_1 \vee A_2$. Therefore, $A \vee A_1 = A_1 \vee A_2 = A \vee A_2$, by the Intersection Theorem. The general result follows by induction on k. The converse is clear.

In view of the above statement, it is now easy to see that if $\mathscr{L}$ is a local Noether lattice which satisfies the weak union condition, then, given primes $P_1, \cdots, P_k$ and an element A such that $A \nleqq P_i$ $(i=1, \cdots, k)$, there exists a principal element $E \leqq A$ such that $E \nleqq P_i$ $(i=1, \cdots, k)$.

The above observations lead to a straightforward generalization of the ring theoretic proof [**6**, p. 406] of the following lemma.

LEMMA. *Let $\mathscr{L}$ be a local Noether lattice which satisfies the weak union condition. And let $A, B \in \mathscr{L}$ such that $A:B=A$. Then there exists a principal element $E \leqq B$ such that $E \nleqq M^2$ and $A:E=A$.*

COROLLARY 5. *Let $\mathscr{L}$ be a Noether lattice which satisfies the weak union condition, and let E be a join-principal element of $\mathscr{L}$. Then either E is a zero divisor (i.e., $EA=0$ for some $A\neq 0 \in \mathscr{L}$) or E is a principal element.*

PROOF. Assume that E is not a zero divisor in $\mathscr{L}$, and let M be any maximal element such that $E \leqq M$. Then $E \nleqq P$ for any prime P of 0 in $\mathscr{L}$, and so $0:\{E\}=0$ in $\mathscr{L}_M$. Since $\mathscr{L}_M$ inherits the weak union condition, by the lemma there is a principal element $\{E_1\} \leqq \{E\}$ such that $0:\{E_1\}=0$ and $\{E_1\} \nleqq \{M\}\{E\}$. Using the technique of the proof of Theorem 1, we can show that $\{E\}$ has a minimal base $\{E_1\}, \cdots, \{E_k\}$ where $\{E_i\}\{E_j\}=0$ for $i\neq j$. But $0:\{E_1\}=0$ implies that $k=1$, so $\{E\}=\{E_1\}$. Since $\{E\}$ is principal in $\mathscr{L}_M$ for every maximal element M in $\mathscr{L}$, we conclude that E is principal in $\mathscr{L}$. Q.E.D.

Using the construction process given earlier in the paper, we can easily see that a Noether lattice which does not satisfy the weak union condition may have many join-principal elements which are neither zero divisors nor principal. If $(\mathscr{L}_i, M_i)$ $(1 \leqq i \leqq n$ and $n \geqq 2)$ are regular local Noether lattices of altitude 1, then any $(A_1, \cdots, A_n) \in M_1/O \oplus \cdots \oplus M_n/O$, $A_i \neq 0$ for each i, is an example of such an element.

Since a meet-principal element of a lattice of ideals L of a local Noetherian ring R is join-irreducible in L, it is a principal ideal in R [**2**].

In general, this is not true for join-principal elements of L. However, since L satisfies the weak union condition and principal elements in L are principal ideals, we have the following corollary.

COROLLARY 6. *Let L be the lattice of ideals of a local Noetherian ring R, and let E be a join-principal element of L. Then E is a zero divisor in L or E is a principal ideal of R.*

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