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ON Q-DENSE AND DENSELY DIVISIBLE LCA GROUPS

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ABSTRACT. In this note the structure of those LCA groups containing a dense subgroup isomorphic to the additive group of rational numbers is investigated. As a result of this investigation, it is shown that an LCA group containing a dense divisible subgroup of finite rank must be divisible.

In [**1**] the authors investigate the structure of those locally compact (Hausdorff) abelian (LCA) groups G containing a dense subgroup of the form $Z(p^\infty)$, where p is a prime. In this note we turn our attention to the following related (and more general) question: Which LCA groups contain as a dense subgroup a homomorphic image of the additive group Q of rational numbers?

All groups considered will be LCA unless otherwise specified. The additive group of the rational numbers is denoted by Q; when considered as a topological group, it has the discrete topology. By R we denote the group of real numbers with its usual topology. If p is a prime we denote by $Z(p^\infty)$ the (discrete) group of p^nth complex roots of unity. The p-adic integers and p-adic numbers are denoted by J_p and F_p respectively, with their usual topologies defined as in [**3**, §10]. Let G be compact and let $\aleph$ be a cardinal number. We write $G^\aleph$ for the (full) direct product of $\aleph$ copies of G with the compact product topology. If G is discrete, we write $G^{\aleph *}$ for the (weak) direct sum of $\aleph$ copies of G, taken discrete. The character group of G is written $\hat{G}$.

Let E denote the minimal divisible extension [**3**, A.15] of the LCA group G. We always topologize E as in [**3**, 25.32(a)], so that E becomes a divisible LCA group containing G as an open subgroup. For example, if $G=\prod J_p$, by which we will always mean the full direct product of the groups J_p, one for each prime p, then E is topologically isomorphic to the local direct product [**3**, 6.16] of the groups F_p relative to the open subgroups J_p, one factor for each p [**2**, 25.32(d)]. We write this group as $\mathrm{LP}(F_p)$. We shall use the fact that $\mathrm{LP}(F_p)$ is self-dual [**3**, 25.34(b)]. We now state a lemma for future reference.

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LEMMA 1. *Let f be a continuous monomorphism from a torsion-free LCA group G into a divisible LCA group D. Then f may be extended to a continuous monomorphism $\bar{f}: E \to D$, where E is the minimal divisible extension of G.*

PROOF. The homomorphism f may be extended to a homomorphism $\bar{f}: E \to D$ by [**3**, A.7]. Continuity of $\bar{f}$ follows from the openness of G in E. That $\bar{f}$ is one-one follows from the fact that E/G is a torsion group [**3**, A.17].

DEFINITION. An LCA group G is called *Q-dense* if and only if there exists a homomorphism $f: Q \to G$ having dense image.

We may reformulate this definition by using the adjoint map [**3**, 24.37]. Thus an LCA group G is Q-dense if and only if there exists a continuous monomorphism f^* mapping $\hat{G}$ into $\hat{Q}$ [**3**, 24.41]. Obvious examples of Q-dense groups are Q and R. By using our second formulation just given, it is easy to show that a compact group is Q-dense if and only if it is solenoidal [**3**, 9.2]. (Here we use the fact that $\hat{Q}$ is algebraically isomorphic to $Q^{\mathfrak{c}*}$, where $\mathfrak{c}$ denotes the power of the continuum.) In particular, $\hat{Q}^{\mathfrak{c}}$ is Q-dense. An example of a totally disconnected Q-dense group other than Q is F_p, and more generally $\mathrm{LP}(F_p)$. In fact, the next lemma shows that the direct product of R, $\hat{Q}^{\mathfrak{c}}$, and $\mathrm{LP}(F_p)$ is also Q-dense. This lemma will lead us to a description of the class of Q-dense groups. The authors are indebted to the referee for suggesting an important simplification in its proof.

LEMMA 2. *The group $R \times \mathrm{LP}(F_p) \times \hat{Q}^{\mathfrak{c}}$ is Q-dense.*

PROOF. We must show that there exists a continuous monomorphism f from $R \times \mathrm{LP}(F_p) \times Q^{\mathfrak{c}*}$ into $\hat{Q}$. First, we set $A = R \times \mathrm{LP}(F_p)$, the group of adèles (see [**2**, §3.1]). Like R, each group F_p contains canonically a dense subgroup isomorphic to the rationals, which we denote by Q in each case. It is shown in [**2**, §3.6] that there exists a continuous homomorphism π from A onto $\hat{Q}$ whose kernel is the subgroup of A consisting of all sequences of the form $(x, x, x, \cdots)$ with $x \in Q$. Let $\varphi: A \to A$ be defined by $\varphi(x_0, x_2, \cdots, x_p, \cdots) = (x_0, 2x_2, \cdots, px_p, \cdots)$. Clearly, φ is a continuous monomorphism. Moreover, we see that $\varphi(A) \cap \ker \pi = \{0\}$; this follows from the fact that no sequence of the form $(n, n, n, \cdots)$, where n is an integer, can belong to $\varphi(A)$, since $n \in pJ_p$ for at most finitely many primes p. Hence the map $\psi = \pi \circ \varphi$ is a continuous monomorphism from A into $\hat{Q}$.

In order to complete our construction of f, it will be necessary to show that the quotient group $\hat{Q}/\psi(A)$ has cardinal number $\mathfrak{c}$. To do this, we first define a homomorphism π' mapping $A/\varphi(A)$ onto $\hat{Q}/\psi(A)$ by the rule

$\pi'(a+\varphi(A))=\pi(a)+\psi(A)$, where $a \in A$. We then observe that ker π' is isomorphic to the countable group Q, and so $\hat{Q}/\psi(A)$ will have cardinality $\mathfrak{c}$ if $A/\varphi(A)$ does also. But the cardinality of $A/\varphi(A)$ certainly exceeds that of $\prod J_p/(\varphi(A)\cap \prod J_p)\cong[\prod J_p/\prod pJ_p]/[(\varphi(A)\cap \prod J_p)/\prod pJ_p]\cong[\prod Z(h)]/H$, where $\prod Z(p)$ is the (uncountable) full direct product of the p-element cyclic groups $Z(p)$ and H is countable. Hence $A/\varphi(A)$ has cardinality $\mathfrak{c}$, and therefore so does $\hat{Q}/\psi(A)$.

Finally, we observe that $\psi(A)$ is a divisible subgroup of $\hat{Q}$. Hence there exists a subgroup D of $\hat{Q}$ such that algebraically, $\hat{Q}\cong\psi(A)\times D$, where $D\cong\hat{Q}/\psi(A)$ [**3**, A.8]. Since D must be divisible, it follows from [**3**, A.14] and the preceding paragraph that $D\cong Q^{\mathfrak{c}^*}$. Combining this isomorphism with the monomorphism $\psi: A\to\psi(A)$ we can easily construct the desired continuous monomorphism f mapping $A\times Q^{\mathfrak{c}^*}$ into $\hat{Q}$. This completes the proof.

We are now able to give a description of the Q-dense LCA groups.

THEOREM 1. *An* LCA *group G is Q-dense if and only if G is topologically isomorphic to Q or to a quotient of $R\times \mathrm{LP}(F_p)\times \hat{Q}^{\mathfrak{c}}$ by a closed subgroup. In particular, every Q-dense* LCA *group is divisible.*

PROOF. If G has the form mentioned, it follows immediately, from Lemma 2 and the fact that a continuous homomorphic image of a Q-dense group is Q-dense, that G is Q-dense.

For the converse, assume that G is Q-dense. Then there exists a continuous monomorphism $f: \hat{G}\to\hat{Q}$. Now $\hat{G}$ can be written in the form $\hat{G}\cong R^n\times\hat{G}_0$, where n is a nonnegative integer and $\hat{G}_0$ has a compact open subgroup [**3**, 24.30]. Now we must have $n\leqq 1$, for if $n>1$ we could find a continuous monomorphism from R^2 into $\hat{Q}$. This would imply the existence, via the adjoint map, of a continuous homomorphism from Q into R^2 having dense image, which is absurd. Next, let K be the (compact) identity component of $\hat{G}_0$. If $K\neq\{0\}$, then $f(K)$ is a nontrivial closed connected subgroup of $\hat{Q}$, and hence $f(K)=\hat{Q}$ (this follows from the fact that all proper quotients of Q are torsion groups). Hence $\hat{G}=K$ and f is a topological isomorphism, so G is just the discrete group Q. On the other hand, if $K=\{0\}$, we have that $\hat{G}_0$ is a totally disconnected torsion-free group, so that its minimal divisible extension E [**3**, 25.32] is also totally disconnected [**3**, 7.8] and torsion-free [**3**, A.16]. We conclude from [**3**, 25.33] that E has the form $Q^{\aleph^*}\times E_0$, where $\aleph$ is a cardinal and E_0 is the minimal divisible extension of a product of groups J_p for various primes p. But since there exists a continuous monomorphism from $\hat{G}_0$ into $\hat{Q}$ there also exists a continuous monomorphism from E into $\hat{Q}$ by Lemma 1. From this it follows that $\aleph\leqq\mathfrak{c}$ and that at most one group J_p appears for each prime p, since, as can be seen by examining the quotients of Q, there

do not exist continuous monomorphisms from J_p^2 into $\hat{Q}$. Hence $\hat{G}_0$ is isomorphic to a closed subgroup of $Q^{c*} \times \mathrm{LP}(F_p)$. This means that $\hat{G}$ is isomorphic to a closed subgroup of $R \times Q^{c*} \times \mathrm{LP}(F_p)$. The proof is completed by dualizing the preceding statement.

REMARK 1. It is not hard to show that all solenoidal groups contain a dense copy of Q. It can also be shown, although we shall not do it here, that all Q-dense LCA groups contain a dense copy of Q, except for the proper quotients of Q taken discrete.

We close with an application of our findings to the study of *densely divisible* groups, that is, groups containing a dense divisible subgroup. If an LCA group G contains a dense divisible subgroup D, we seek conditions under which we may conclude that G is itself divisible. One sufficient condition is that either G or $\hat{G}$ be compactly generated; this can be shown with the aid of the structure theorem for compactly generated LCA groups [**3**, 9.8]. Below we show that, if D is not "too large", the same conclusion can be drawn.

THEOREM 2. *Let G be an* LCA *group containing a dense divisible subgroup of finite rank. Then G is divisible.*

PROOF. We first recall that a divisible group of rank n can be written as the direct sum of n groups $D_1, \cdots, D_n$, where each D_i is isomorphic either to Q or $Z(p^\infty)$ for some prime p [**3**, A.14]. Our proof proceeds by induction. If G is an LCA group containing a dense divisible subgroup of rank 1, then G is Q-dense (recall that $Z(p^\infty)$ is a quotient of Q), so that G is divisible by Theorem 1. Next, assume that any LCA group containing a dense divisible subgroup of rank n is divisible. Now let G be an LCA group containing a dense divisible subgroup D of rank $n+1$. We wish to show that G is divisible. Write $D = D_1 \oplus \cdots \oplus D_n \oplus D_{n+1}$ as above, and set $D_* = D_1 \oplus \cdots \oplus D_n$. The closure $\bar{D}_*$ of D_* in G is, by the inductive assumption, a divisible subgroup of G. If $\bar{D}_* = G$ there is nothing to prove. Otherwise, it is sufficient to show that the quotient $G/\bar{D}_*$ is divisible, for then it follows immediately that G is divisible. But if π denotes the natural map from G onto $G/\bar{D}_*$, it is easy to see that $\pi(D_{n+1})$ is dense in $G/\bar{D}_*$, so that $G/\bar{D}_*$ is Q-dense and hence divisible. This completes the proof.

REMARK 2. Theorem 2 no longer remains valid if "finite" is replaced by "countable". To see this, let E be the minimal divisible extension of the group $J_p^{\aleph_0}$, where p is a prime; this group is described in [**3**, 25.32(c)] and [**4**, 2.2]. Since there exist continuous monomorphisms from J_p into $\hat{Q}$, we can construct a continuous monomorphism from $J_p^{\aleph_0}$ into $\hat{Q}^{\aleph_0}$. This monomorphism may be extended to a continuous monomorphism $f: E \to \hat{Q}^{\aleph_0}$ by Lemma 1. Hence the adjoint map $f^*: Q_0^{\aleph_0*} \to \hat{E}$ has dense image

[**3**, 24.41]. Thus $\hat{E}$ contains a dense divisible subgroup of countable rank, but $\hat{E}$ is not divisible by [**4**, 2.7].

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