INFINITE MATRICES AND INVARIANT MEANS

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Abstract. Let $\sigma$ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^p(n) \neq n$ for all positive integers $n$ and $p$, where $\sigma^p(n) = \sigma(\sigma^{p-1}(n))$, $p = 1, 2, \cdots$. A continuous linear functional $\varphi$ on the space of real bounded sequences is an invariant mean if $\varphi(x) \geq 0$ when the sequence $x = \{x_n\}$ has $x_n \geq 0$ for all $n$, $\varphi((1, 1, 1, \cdots)) = +1$, and $\varphi((x_{\sigma(n)})) = \varphi(x)$ for all bounded sequences $x$. Let $V_\sigma$ be the set of bounded sequences all of whose invariant means are equal. If $A = (a_{nk})$ is a real infinite matrix, then $A$ is said to be (1) $\sigma$-conservative if $Ax = \{\sum_k a_{nk}x_k\} \in V_\sigma$ for all convergent sequences $x$, (2) $\sigma$-regular if $Ax \in V_\sigma$ and $\varphi(Ax) = \lim x$ for all convergent sequences $x$ and all invariant means $\varphi$, and (3) $\sigma$-coercive if $Ax \in V_\sigma$ for all bounded sequences $x$. Necessary and sufficient conditions are obtained to characterize these classes of matrices.

1. Introduction. Let $\sigma$ be a mapping of the set of positive integers into itself. A continuous linear functional $\varphi$ on $m$, the space of real bounded sequences, is said to be an invariant mean or a $\sigma$-mean if and only if (1) $\varphi(x) \geq 0$ when the sequence $x = \{x_n\}$ has $x_n \geq 0$ for all $n$, (2) $\varphi(e) = 1$, where $e = \{1, 1, 1, \cdots\}$, and (3) $\varphi((x_{\sigma(n)})) = \varphi(x)$ for all $x \in m$. For certain kinds of mappings $\sigma$, every invariant mean $\varphi$ extends the limit functional on the space $c$ of real convergent sequences, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. Consequently, $c \subseteq V_\sigma$ where $V_\sigma$ is the set of bounded sequences all of whose $\sigma$-means are equal.

When $\sigma(n) = n+1$, the $\sigma$-means are the classical Banach limits on $m$ and $V_\sigma$ is the set of almost convergent sequences [5]. If $A = (a_{nk})$ is an infinite matrix with real entries such that $Ax = \{\sum_k a_{nk}x_k\}$ is an almost convergent sequence for every convergent sequence $x$, $A$ is said to be an almost conservative matrix [4]. When the common value of all Banach limits of $Ax$ is $\lim x$ for all $x \in c$, then the almost conservative matrix $A$ is said to be almost regular. J. P. King [4] gave necessary and sufficient conditions that a matrix be almost conservative or almost regular. More
recently, Eizen and Laush [2] considered the class of almost coercive matrices, those for which \( Ax \) is almost convergent for every bounded sequence \( x \). In this paper we define analogous notions of \( \sigma \)-conservative, \( \sigma \)-regular, and \( \sigma \)-coercive matrices and obtain conditions which characterize them.

2. Preliminaries. We consider the spaces \( c \) and \( m \) as Banach spaces normed by \( \| x \| = \sup \{|x_n|\} \). Let \( c' \) and \( m' \) denote the conjugate spaces of \( c \) and \( m \) respectively, normed in the usual way. It is well known that each \( f \in c' \) has the representation

\[
f(x) = \left( \lim x \right) f(e) + \sum_{k=1}^{\infty} x_k f(e^k),
\]

where \( x = \{x_k\} \) and \( e^k \) is the sequence having \(+1\) in its \( k\)th entry and zeros elsewhere. Furthermore, \( \| f \| \) is given by \( |f(e) - \sum_{k=1}^{\infty} f(e^k)| + \sum_{k=1}^{\infty} |f(e^k)| \).

The set \( \{e, e^1, e^2, \cdots\} \) is a Schauder basis for \( c \), and every \( x = \{x_k\} \in c \) can be written uniquely as \( x = (\lim x)e + \sum_k (x_k - \lim x)e^k \) [8].

Throughout this paper we deal only with mappings \( \sigma \) of the set of positive integers into itself which are one-to-one and are such that \( \sigma^p(n) \neq n \) for all positive integers \( n \) and \( p \), where \( \sigma^p(n) \) denotes the \( p\)th iterate of the mapping \( \sigma \) at \( n \). For such mappings, every \( \sigma \)-mean extends the limit functional on \( c \) [6].

If \( x = \{x_n\} \), set \( Tx = \{x_{\sigma(n)}\} \). It can be shown that the set \( V_\sigma \) described in the Introduction can be characterized as the set of all bounded sequences \( x \) for which \( \lim_{p} (x + T x + \cdots + T^p x)/(p+1) \) exists in the space \( m \) and has the form \( Le, L \) being the common value of all \( \sigma \)-means at \( x \) [6]. We write \( L = \sigma-lim x \).

3. \( \sigma \)-conservative and \( \sigma \)-regular matrices. All matrices in this paper are real infinite matrices. For such matrices, the notions of being almost conservative and almost regular can be generalized as follows.

**Definition 1.** An infinite matrix \( A \) is said to be \( \sigma \)-conservative if and only if \( Ax = \{ \sum_k a_{nk}x_k \} \in V_\sigma \) for all \( x \in c \).

**Definition 2.** An infinite matrix \( A \) is said to be \( \sigma \)-regular if and only if it is \( \sigma \)-conservative and \( \sigma-lim Ax = \lim x \) for all \( x \in c \).

**Theorem 1.** The matrix \( A \) is \( \sigma \)-conservative if and only if

1. \( \| A \| = \sup_n \{ \sum_k |a_{nk}| \} < + \infty \),
2. \( a_{(k)} = \{a_{nk}\}_{k=1}^{\infty} \in V_\sigma \) for each \( k \), and
3. \( a = \{ \sum_k a_{nk} \}_{k=1}^{\infty} \in V_\sigma \).

When \( A \) is \( \sigma \)-conservative, the \( \sigma \)-limit of \( Ax \) is \( (\lim x)[u - \sum_k u_k] + \sum_k x_k u_k \) for every \( x = \{x_k\} \in c \), where \( u = \sigma-lim a \) and \( u_k = \sigma-lim a_{(k)} \), \( k = 1, 2, \cdots \).
THEOREM 2. The matrix $A$ is $\sigma$-regular if and only if

(1) $\|A\| < +\infty$,
(2) $a_{(k)} \in V_\sigma$ with $\sigma$-limit zero for each $k$, and
(3) $a \in V_\sigma$ with $\sigma$-limit $+1$.

For typographical convenience we shall use the notation $a(n, k)$ to denote the element $a_{nk}$ of the matrix $A$ in the following proofs.

PROOF OF THEOREM 1. Let us first suppose that conditions (1), (2) and (3) hold. Let $p$ be any nonnegative integer and let $x \in c$. We have

$$(Ax + TAx + \cdots + T^pAx)/(p + 1) = \left( \sum_k [a(n, k) + a(\sigma(n), k) + \cdots + a(\sigma^p(n), k)]x_k/(p + 1) \right)_n.$$ 

For every positive integer $n$, set

$$t_{pn}(x) = \sum_{k=1}^{\infty} \sum_{j=0}^{p} a(\sigma^j(n), k)x_k/(p + 1).$$

Then we have

$$|t_{pn}(x)| \leq \sum_{k=1}^{\infty} \sum_{j=0}^{p} |a(\sigma^j(n), k)| \cdot |x_k|/(p + 1) \leq \|x\|/(p + 1) \cdot \left[ \sum_{j=0}^{p} \sum_{k=1}^{\infty} |a(\sigma^j(n), k)| \right] \leq \|A\| \cdot \|x\|.$$ 

Since $t_{pn}$ is obviously linear on $c$, it follows that $t_{pn} \in c'$ and that

$$\|t_{pn}\| \leq \|A\|.$$ 

Now,

$$t_{pn}(e) = \left[ \sum_{k=1}^{\infty} \sum_{j=0}^{p} a(\sigma^j(n), k) \right]/(p + 1) = \left[ \sum_{j=0}^{p} \sum_{k=1}^{\infty} a(\sigma^j(n), k) \right]/(p + 1),$$

so $\lim_p t_{pn}(e)$ exists uniformly in $n$ and equals $u$, the $\sigma$-limit of $a$, since $a \in V_\sigma$. Similarly, $\lim_p t_{pn}(e^k) = u_k$, the $\sigma$-limit of $a_{(k)}$ for each $k$, uniformly in $n$. Since $\{e, e^1, e^2, \cdots\}$ is a fundamental set in $c$, and $\sup_p \{|t_{pn}(x)|\}$ is finite for each $x \in c$, it follows that $\lim_p t_{pn}(x) = t_n(x)$ exists for all $x \in c$ [1, p. 60]. Furthermore, $\|t_n\| \leq \lim inf_p \|t_{pn}\| \leq \|A\|$ for each $n$, and $t_n \in c'$.

Thus,

$$t_n(x) = \lim x \left[ t_n(e) - \sum_k t_n(e^k) \right] + \sum_k x_k t_n(e^k) = \lim x \left[ u - \sum_k u_k \right] + \sum_k x_k u_k,$$

an expression independent of $n$. Denote this expression by $L(x)$. 

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In order to see that \( \lim_{n \to \infty} t_{pn}(x) = L(x) \) uniformly in \( n \), set \( F_{pn}(x) = t_{pn}(x) - L(x) \). Then \( F_{pn} \in c' \), \( \|F_{pn}\| \leq 2\|A\| \) for all \( p \) and \( n \), \( \lim_{p} F_{pn}(e) = 0 \) uniformly in \( n \), and \( \lim_{p} F_{pn}(e^{k}) = 0 \) uniformly in \( n \) for each \( k \). Let \( K \) be an arbitrary positive integer. Then

\[
x = (\lim x)e + \sum_{k=1}^{K} (x_{k} - \lim x)e^{k} + \sum_{k=K+1}^{\infty} (x_{k} - \lim x)e^{k},
\]

so we have

\[
F_{pn}(x) = (\lim x)F_{pn}(e) + \sum_{k=1}^{K} (x_{k} - \lim x)F_{pn}(e^{k})
\]

\[
+ F_{pn}\left( \sum_{k=K+1}^{\infty} (x_{k} - \lim x)e^{k} \right).
\]

Now,

\[
\left| F_{pn}\left( \sum_{k=K+1}^{\infty} (x_{k} - \lim x)e^{k} \right) \right| \leq 2\|A\| \cdot \sup_{k \geq K+1} \{|x_{k} - \lim x|\}
\]

for all \( p \) and \( n \). By first choosing a fixed \( K \) large enough, it is easy to see that each of the three displayed terms for \( F_{pn}(x) \) can be made to be uniformly small in absolute value for all sufficiently large \( p \), so \( \lim_{p} F_{pn}(x) = 0 \) uniformly in \( n \). This shows that

\[
\lim_{p}(Ax + TAx + \cdots + T^{p}Ax)/(p + 1) = L(x)e,
\]

so that \( Ax \in V_{\sigma} \) and the matrix \( A \) is \( \sigma \)-conservative.

Conversely, suppose that \( A \) is \( \sigma \)-conservative. If \( x \) is any null sequence, then \( Ax \in V_{\sigma} \subset m \). It follows from the proof of [3, Theorem 1, pp. 45 and 46] that \( \|A\| < +\infty \). Furthermore, since \( Ae = a \) and \( A e^{k} = a^{(k)} \), the other two conditions are necessary for \( \sigma \)-conservative matrices.

**Proof of Theorem 2.** If a matrix \( A \) satisfies the three conditions of the theorem, then it is a \( \sigma \)-conservative matrix. For \( x \in c \), the \( \sigma \)-limit of \( Ax \) is \( L(x) \), which reduces to \( \lim x \), since \( u = 1 \) and \( u_{k} = 0 \) for each \( k \). Hence, \( A \) is a \( \sigma \)-regular matrix. Conversely, if \( A \) is \( \sigma \)-regular, then \( \sigma \)-\( \lim Ae = +1 = \sigma \)-\( \lim a \), \( \sigma \)-\( \lim A e^{k} = 0 = \sigma \)-\( \lim a^{(k)} \), and \( \|A\| \) is finite, as in the proof of Theorem 1.

4. \( \sigma \)-coercive matrices.

**Definition 3.** A matrix \( A \) is \( \sigma \)-coercive if and only if \( Ax \in V_{\sigma} \) for all \( x \in m \).

**Theorem 3.** The matrix \( A \) is \( \sigma \)-coercive if and only if

1. \( \|A\| \) is finite,
2. \( a^{(k)} \in V_{\sigma} \) for each \( k \), and
3. \( \lim_{p} \sum_{k=1}^{p} |\sum_{j=0}^{p} [a(\sigma^{j}(n), k) - u_{k}]|/(p + 1) = 0 \) uniformly in \( n \), where
Proof. Let us first assume that the matrix $A$ satisfies conditions (1), (2) and (3). For any positive integer $K$,

$$
\sum_{k=1}^{K} |u_k| = \sum_{k=1}^{K} \lim_{p \to \infty} \left| \sum_{j=0}^{p} a(\sigma'(n), k) \right| / (p + 1)
$$

$$
= \lim_{p \to \infty} \sum_{k=1}^{K} \left| \sum_{j=0}^{p} a(\sigma'(n), k) \right| / (p + 1)
$$

$$
\leq \lim sup \sum_{p} \sum_{j=0}^{\infty} |a(\sigma'(n), k)| / (p + 1) \leq \|A\|.
$$

This shows that $\sum_{k=1}^{\infty} |u_k|$ converges, and that $\sum_k u_k x_k$ is defined for every bounded sequence $x = \{x_k\}$.

Let $x$ be an arbitrary bounded sequence. For every positive integer $p$,

$$
(Ax + TAx + \cdots + T^nAx)/(p + 1) = \left( \sum_k u_k x_k \right) e
$$

so

$$
\left\| (Ax + TAx + \cdots + T^nAx)/(p + 1) - \left( \sum_k u_k x_k \right) e \right\|
$$

$$
= \sup_{n} \left\| \sum_{k=1}^{\infty} \left( \sum_{j=0}^{p} [a(\sigma'(n), k) - u_k] / (p + 1) \right) x_k \right\|
$$

$$
\leq \|x\| \cdot \sup_{n} \sum_{k=1}^{\infty} \sum_{j=0}^{p} |a(\sigma'(n), k) - u_k| / (p + 1).
$$

Let $p \to \infty$. By the uniformity of the limits in condition (3), it follows that $(Ax + TAx + \cdots + T^nAx)/(p + 1) \to \left( \sum_k u_k x_k \right) e$, and that $Ax \in V_\sigma$ with $\sigma$-limit $\sum_k u_k x_k$.

Next, suppose that $A$ is a $\sigma$-coercive matrix. Then, since $A$ is $\sigma$-conservative, we have conditions (1) and (2) from Theorem 1. In order to see that (3) holds, we proceed as in [2], by first showing that the limit in question is zero for each $n$, and secondly showing that the limit is uniform in $n$.

Thus, suppose that for some $n$, we have

$$
\lim sup \sum_{p} \sum_{j=0}^{\infty} |a(\sigma'(n), k) - u_k| / (p + 1) = N > 0.
$$
Since $\|A\|$ is finite, $N$ is finite also. We observe that since $\sum |u_k| < +\infty$, the matrix $B=(b_{nk})$, where $b_{nk}=a_{nk}-u_k$, is also a $\sigma$-coercive matrix. If one sets $F_{k,p} = \sum_{j=0}^{p} [a(\sigma^j(n), k) - u_k]/(p+1)$, and $E_{k,p} = F_{k,p}$, one can follow the construction in the proof of Theorem 2.1 in [2] to obtain a bounded sequence whose transform by the matrix $B$ is not in $V_\sigma$. This contradiction shows that the limit in (3) is zero for every $n$.

To show that this convergence is uniform in $n$, we invoke the following lemma, which is proved in [7].

**Lemma.** Let $\{H(n)\}$ be a countable family of matrices $H(n)=(h_{pk}(n))$ such that $\|H(n)\| \leq M < +\infty$ for all $n$ and $\lim_p h_{pk}(n)=0$ for each $k$, uniformly in $n$. Then $\lim_p \sum_k h_{pk}(n)x_k=0$ uniformly in $n$ for all $x \in m$ if and only if $\lim_p \sum_k |h_{pk}(n)|=0$ uniformly in $n$.

We let $h_{pk}(n)=\sum_{j=0}^{p} [a(\sigma^j(n), k) - u_k]/(p+1)$ and let $H(n)$ be the matrix $(h_{pk}(n))$. It is easy to see that $\|H(n)\| \leq 2\|A\|$ for every $n$, and that $\lim_p h_{pk}(n)=0$ for each $k$, uniformly in $n$ by condition (2). For any $x \in m$, $\lim_p \sum_k h_{pk}(n)x_k=\sigma$-lim $Ax-\sum_k u_kx_k$, and the limit exists uniformly in $n$ since $Ax \in V_\sigma$. Moreover this limit is zero since

$$\left| \sum_k h_{pk}(n)x_k \right| \leq \|x\| \cdot \sum_k \left| \sum_{j=0}^{p} [a(\sigma^j(n), k) - u_k] \right|/(p+1).$$

Thus, $\lim_p \sum_k |h_{pk}(n)|=0$ uniformly in $n$, and the matrix $A$ satisfies condition (3).

**Theorem 4.** The classes of $\sigma$-regular and $\sigma$-coercive matrices are disjoint.

**Proof.** If $A$ were a $\sigma$-regular and a $\sigma$-conservative matrix, then $\sigma$-lim $a_{(k)}=0=u_k$ for every $k$. The conditions $\sigma$-lim $a=+1$ and

$$\lim_p \sum_k \left| \sum_{j=0}^{p} a(\sigma^j(n), k) \right| = 0$$

are incompatible, since

$$\left| \frac{1}{p+1} \sum_{j=0}^{p} \sum_k a(\sigma^j(n), k) \right| = \left| \sum_k \frac{1}{p+1} \sum_{j=0}^{p} a(\sigma^j(n), k) \right|$$

$$\leq \sum_k \frac{1}{p+1} \left| \sum_{j=0}^{p} a(\sigma^j(n), k) \right|.$$
References


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