SOME OPERATOR MONOTONE FUNCTIONS

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ABSTRACT. A short proof is given based on C*-algebra theory for the well-known theorem that if $S$ and $T$ are bounded selfadjoint operators on a Hilbert space such that $0 \leq S \leq T$ then $S^\alpha \leq T^\alpha$ for each $0 \leq \alpha \leq 1$.

THEOREM. If $S$ and $T$ are bounded selfadjoint operators on a Hilbert space $H$ such that $0 \leq S \leq T$ then $S^\alpha \leq T^\alpha$ for each $\alpha$ in the interval $[0, 1]$.

REMARK. The theorem says that each function $t \rightarrow t^\alpha$, with $0 \leq \alpha \leq 1$, is operator monotone on the set of positive operators in $B(H)$. This was first proved by K. Löwner, who gave a complete description of operator monotone functions. Later T. Ogasawara gave a short proof of the operator monotonicity for the square root function. We present here a simple proof based on C*-algebra theory.

Proof. If $0 \leq S \leq T$ then $S + \varepsilon I \leq T + \varepsilon I$ for each $\varepsilon > 0$; and $S + \varepsilon I$ and $T + \varepsilon I$ are both invertible. Since $(S + \varepsilon I)^\alpha$ converges to $S^\alpha$ in norm when $\varepsilon \rightarrow 0$ for each $\alpha > 0$, and since the positive operators in $B(H)$ form a norm closed set, it suffices to prove the theorem assuming that $S$ and $T$ are invertible. (The case $\alpha = 0$ can be verified directly, since $S^0$ is the range projection of $S$.)

Let $E$ denote the set of exponents $\alpha$ in $[0, 1]$ for which the function $t \rightarrow t^\alpha$ is operator monotone. Trivially $0 \in E$ and $1 \in E$. Since the function $\alpha \rightarrow S^\alpha$ is continuous from $[0, 1]$ to $B(H)$ in the norm topology we see that $E$ is a closed set. The proof will be complete when we show that $E$ is convex.

Take $\alpha$ and $\beta$ in $E$. Then $S^\alpha \leq T^\alpha$; hence $T^{-\alpha/2}S^\alpha T^{-\alpha/2} \leq I$. It follows that $\|S^{\alpha/2}T^{-\alpha/2}\| \leq 1$. Similarly $\|S^{\beta/2}T^{-\beta/2}\| \leq 1$. With $\rho(A)$ the spectral radius of an operator $A$ we have $\rho(AB) = \rho(BA)$. Therefore

\[
\rho(T^{-(\alpha+\beta)/4}S^{(\alpha+\beta)/2}T^{-(\alpha+\beta)/4}) = \rho(T^{-(\alpha-\beta)/4}T^{-(\alpha+\beta)/4}S^{(\alpha+\beta)/2}T^{-(\alpha-\beta)/4}T^{-(\alpha+\beta)/4}) = \rho(T^{-\alpha/2}S^{(\alpha+\beta)/2}T^{-\alpha/2}) \leq \|T^{-\beta/2}S^{(\alpha+\beta)/2}T^{-\alpha/2}\| \leq 1.
\]

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309
It follows that $T^{-(a+b)/4} S^{(a+b)/2} T^{-(a+b)/4} \leq I$, so that $S^{(a+b)/2} \leq T^{(a+b)/2}$. This shows that $(a+b)/2 \in E$ which completes the proof.

REFERENCES


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