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SOME OPERATOR MONOTONE FUNCTIONS[1]

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ABSTRACT. A short proof is given based on C^*-algebra theory for the well-known theorem that if S and T are bounded selfadjoint operators on a Hilbert space such that $0 \leqq S \leqq T$ then $S^\alpha \leqq T^\alpha$ for each $0 \leqq \alpha \leqq 1$.

THEOREM. *If S and T are bounded selfadjoint operators on a Hilbert space $\mathfrak{H}$ such that $0 \leqq S \leqq T$ then $S^\alpha \leqq T^\alpha$ for each α in the interval* $[0, 1]$.

REMARK. The theorem says that each function $t \rightarrow t^\alpha$, with $0 \leqq \alpha \leqq 1$, is operator monotone on the set of positive operators in $B(\mathfrak{H})$. This was first proved by K. Löwner, who gave a complete description of operator monotone functions. Later T. Ogasawara gave a short proof of the operator monotonicity for the square root function. We present here a simple proof based on C^*-algebra theory.

PROOF. If $0 \leqq S \leqq T$ then $S + \varepsilon I \leqq T + \varepsilon I$ for each $\varepsilon > 0$; and $S + \varepsilon I$ and $T + \varepsilon I$ are both invertible. Since $(S + \varepsilon I)^\alpha$ converges to S^α in norm when $\varepsilon \rightarrow 0$ for each $\alpha > 0$, and since the positive operators in $B(\mathfrak{H})$ form a norm closed set, it suffices to prove the theorem assuming that S and T are invertible. (The case $\alpha = 0$ can be verified directly, since S^0 is the range projection of S.)

Let E denote the set of exponents α in $[0, 1]$ for which the function $t \rightarrow t^\alpha$ is operator monotone. Trivially $0 \in E$ and $1 \in E$. Since the function $\alpha \rightarrow S^\alpha$ is continuous from $[0, 1]$ to $B(\mathfrak{H})$ in the norm topology we see that E is a closed set. The proof will be complete when we show that E is convex.

Take α and β in E. Then $S^\alpha \leqq T^\alpha$; hence $T^{-\alpha/2} S^\alpha T^{-\alpha/2} \leqq I$. It follows that $\|S^{\alpha/2} T^{-\alpha/2}\| \leqq 1$. Similarly $\|S^{\beta/2} T^{-\beta/2}\| \leqq 1$. With $\rho(A)$ the spectral radius of an operator A we have $\rho(AB) = \rho(BA)$. Therefore

$$\begin{aligned} \rho(T^{-(\alpha+\beta)/4} S^{(\alpha+\beta)/2} T^{-(\alpha+\beta)/4}) &= \rho(T^{(\alpha-\beta)/4} T^{-(\alpha+\beta)/4} S^{(\alpha+\beta)/2} T^{-(\alpha+\beta)/4} T^{-(\alpha-\beta)/4}) \\ &= \rho(T^{-\beta/2} S^{(\alpha+\beta)/2} T^{-\alpha/2}) \leqq \|T^{-\beta/2} S^{(\alpha+\beta)/2} T^{-\alpha/2}\| \\ &\leqq \|T^{-\beta/2} S^{\beta/2}\| \, \|S^{\alpha/2} T^{-\alpha/2}\| \leqq 1. \end{aligned}$$

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It follows that $T^{-(\alpha+\beta)/4}S^{(\alpha+\beta)/2}T^{-(\alpha+\beta)/4} \leqq I$, so that $S^{(\alpha+\beta)/2} \leqq T^{(\alpha+\beta)/2}$. This shows that $(\alpha+\beta)/2 \in E$ which completes the proof.

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