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FIXED POINT THEOREMS FOR LIPSCHITZIAN PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. Let X be a Banach space and $D\subset X$. A mapping $U: D\to X$ is said to be pseudo-contractive if, for all $u, v\in D$ and all $r>0$, $\|u-v\|\leqq\|(1+r)(u-v)-r(U(u)-U(v))\|$. A recent fixed point theorem of W. V. Petryshyn is used to prove: If G is an open bounded subset of X with $0\in G$ and $U:\bar{G}\to X$ is a lipschitzian pseudo-contractive mapping satisfying (i) $U(x)\neq\lambda x$ for $x\in\partial G$, $\lambda>1$, and (ii) $(I-U)(\bar{G})$ is closed, then U has a fixed point in $\bar{G}$. This result yields fixed point theorems for pseudo-contractive mappings in uniformly convex spaces and for "strongly" pseudo-contractive mappings in reflexive spaces.

Let X be a Banach space and $D\subset X$. A mapping $U: D\to X$ is said to be *pseudo-contractive* if, for all $u, v\in D$ and all $r>0$,

$$\|u - v\| \leqq \|(1 + r)(u - v) - r(U(u) - U(v))\|.$$

A characterization of F. Browder given in [1] establishes the importance of this class of mappings; he observes that a necessary and sufficient condition that $U: D\to X$ be pseudo-contractive is that $I-U$ be accretive.

In this paper we prove a fixed point theorem for such mappings by imposing the Leray-Schauder condition ((i) below) used by Browder [2] in his study of semicontractive mappings. Our results illustrate the firm connection which exists between the lipschitzian pseudo-contractive mappings and the class of nonexpansive mappings. This connection was observed by Kirk in [3], and Theorem 2 below represents a considerable generalization of Theorem 1 of [3] in a slightly more restricted setting.

Throughout the paper we use ∂A to denote the boundary of a set $A\subset X$.

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THEOREM 1. *Let X be a Banach space, G an open bounded subset of X with $0 \in G$, and let $U:\bar{G}\to X$ be a lipschitzian pseudo-contractive mapping satisfying*:

(i) $U(x)\neq\lambda x$ *if* $x \in \partial G$ *and* $\lambda>1$,

(ii) $(I-U)(\bar{G})$ *is closed.*

Then U has a fixed point in $\bar{G}$.

THEOREM 2. *Let X be a uniformly convex Banach space whose conjugate space X^* is also uniformly convex, let G be a bounded open convex subset of X with $0 \in G$, and let $U:X\to X$ be a lipschitzian pseudo-contractive mapping satisfying $U(x)\neq\lambda x$ if $x \in \partial G$ and $\lambda>1$. Then U has a fixed point in $\bar{G}$.*

Under the assumptions of Theorem 2, $I-U$ is a continuous accretive mapping defined on all of X, so by Theorem 4 of Browder [**2**], $I-U$ is demiclosed. Since $\bar{G}$ is weakly closed, $(I-U)(\bar{G})$ is closed and Theorem 2 follows from Theorem 1.

Our proof of Theorem 1 makes use of a recent result of W. V. Petryshyn [**6**, Theorem 7]. With G as in Theorem 1, he has shown that if $T:\bar{G}\to X$ is a 1-set-contraction (defined in the remark below) satisfying (i) and (ii), then T has a fixed point in $\bar{G}$. Nonexpansive mappings, or more generally the semicontractive mappings of Browder [**2**], provide important examples of 1-set-contractions. We will apply Petryshyn's result below in cases where either T is a contraction mapping, or T is nonexpansive. Its full generality shows that Theorem 1 holds for the lipschitzian "1-set pseudo-contractive mappings" as indicated in our final remark.

PROOF OF THEOREM 1. Let $0<r<1$ be chosen so that rU is a contraction mapping. Define mappings S, T of $\bar{G}$ into X by $S=(1-r)I$, $T=I-rU$. Then T is one-to-one, $T(G)$ is open, $\partial T(G)=T(\partial G)$, and thus $T(\bar{G})=\mathrm{cl}(T(G))$. Since rU satisfies (i) and (ii) on $\bar{G}$, by Petryshyn's Theorem there exists $x \in \bar{G}$ such that $x=rU(x)$. Hence $x \in G$ (because U satisfies (i)) and so $0=T(x) \in T(G)$ yielding $0 \in \mathrm{int}\, B$ where $B=\mathrm{cl}(T(G))$.

Since U is pseudo-contractive, for each $x, y \in G$,

$$\begin{aligned}\|x-y\| &\leqq \|(1+r)(x-y)-r(U(x)-U(y))\| \\ &\leqq \|(x-rU(x))-(y-rU(y))\|+r\,\|x-y\|;\end{aligned}$$

thus $(1-r)\|x-y\|\leqq\|T(x)-T(y)\|$ which yields

$$\|S(x)-S(y)\|\leqq\|T(x)-T(y)\|,\qquad x,y\in G.$$

Now define $H:B\to X$ by $H(z)=ST^{-1}(z)$. Then if $z_1, z_2 \in B$,

$$\begin{aligned}\|H(z_1)-H(z_2)\| &= \|ST^{-1}(z_1)-ST^{-1}(z_2)\| \\ &\leqq \|TT^{-1}(z_1)-TT^{-1}(z_2)\| = \|z_1-z_2\|,\end{aligned}$$

so H is nonexpansive on B.

To see that $(I-H)(B)$ is closed, suppose $z_n-H(z_n)\to y$, $z_n\in B$. Then $z_n-(1-r)T^{-1}(z_n)\to y$ yielding

$$z_n/(1-r)-T^{-1}(z_n)\to y/(1-r).$$

Let $z=y/(1-r)$ and let $x_n=T^{-1}(z_n)$. Then

$$\begin{aligned} r[x_n-U(x_n)]/(1-r) &= [x_n-rU(x_n)]/(1-r)-x_n \\ &= T(x_n)/(1-r)-x_n\to z, \end{aligned}$$

and thus $x_n-U(x_n)\to(1-r)z/r$. Since $(I-U)(\bar G)$ is closed, there exists $x\in\bar G$ such that $x-U(x)=(1-r)z/r$. Then

$$(1-r)z=r(x-U(x))=x-rU(x)-(1-r)x=T(x)-(1-r)x,$$

yielding $T(x)/(1-r)-x=z$. Letting $w=T(x)$ we have $w/(1-r)-T^{-1}(w)=z$, so $w-(1-r)T^{-1}(w)=(1-r)z=y$. Hence $w-H(w)=y$ and we conclude $(I-H)(B)$ is closed.

Now we show that H satisfies (i) on B. Let $x\in\partial B$ and suppose $H(x)=\lambda x$ for some $\lambda>1$. Then $T^{-1}(x)=\lambda x/(1-r)$, and since $T(\partial G)=\partial T(G)$, we conclude $\lambda x/(1-r)\in\partial G$. Thus we have $x=T(\lambda x/(1-r))$, so

$$x=\lambda x/(1-r)-rU(\lambda x/(1-r)),$$

which implies $U(\lambda x/(1-r))=(\lambda+r-1)x/r(1-r)$.

Let $\bar x=\lambda x/(1-r)$. Then $\bar x\in\partial G$ and $U(\bar x)=\mu\bar x$ where $\mu=(\lambda+r-1)/\lambda r$. But $\mu-1=(\lambda-1)(1-r)/\lambda r>0$, and this contradicts our hypothesis (i) for U on ∂G.

Therefore, we conclude that H satisfies all the hypotheses for Petryshyn's theorem on B, so there exists $y\in B$ such that $H(y)=y$. Letting $x=T^{-1}(y)$, $S(x)=ST^{-1}(y)=H(y)=y=T(x)$; thus $(1-r)x=x-rU(x)$ and we have $U(x)=x$ completing the proof.

We say that a mapping $U:X\to X$ *is strongly pseudo-contractive relative to* $D\subset X$ if for each $x\in X$ and $r>0$ there exists a number $\alpha_r(x)<1$ such that

$$\|x-y\|\leqq\alpha_r(x)\,\|(1+r)(x-y)-r(U(x)-U(y))\|,\qquad y\in D.$$

The following is another consequence of Theorem 1.

THEOREM 3. *Let X be a reflexive Banach space, G a bounded open convex subset of X with $0\in G$, and suppose $U:X\to X$ is a lipschitzian strongly pseudo-contractive mapping relative to $\bar G$ satisfying*

(i) $U(x)\neq\lambda x$ *if* $x\in\partial G$, $\lambda>1$.

Then U has a fixed point in $\bar G$.

PROOF. In view of Theorem 1 we need only show that $(I-U)(\bar{G})$ is closed, and because X is reflexive we need only show that $I-U$ is demiclosed. Thus if $u_j \rightharpoonup u_0$ weakly and if $(I-U)(u_j) \to w$ strongly then we must show $(I-U)(u_0)=w$. Defining $F: X \to X$ by $F(x)=U(x)+w$, then F is lipschitzian and strongly pseudo-contractive on X relative to $\bar{G}$, and furthermore $(I-F)u_j \to 0$ strongly. We show that this implies $u_j \to u_0$ strongly and this, with continuity of U, gives the desired result.

Choose $\lambda>0$ so small that λF is a contraction mapping with lipschitz constant $\lambda_0<1$, and let $r>0$ satisfy $\lambda=r/(r+1)$. Since F is strongly pseudo-contractive on X relative to $\bar{G}$, for each $x \in X$ there exists $\alpha(x)$ such that

$$\|x-y\| \leqq \alpha(x) \|(1+r)(x-y)-r(F(x)-F(y))\|, \qquad y \in \bar{G};$$

this is equivalent to

$$(1-\lambda)\|x-y\| \leqq \alpha(x)\|x-y-\lambda(F(x)-F(y))\|,$$

and since $(I-\lambda F)(X)=X$, we see that the mapping $F_\lambda=I-\lambda F$ satisfies

$$(*) \quad \|(1-\lambda)F_\lambda^{-1}(x)-(1-\lambda)F_\lambda^{-1}(y)\| \leqq \alpha(x)\|x-y\|, \qquad x \in X,\ y \in \bar{G}.$$

Now let $z_j=F_\lambda u_j$. Then

$$\begin{aligned} z_j-(1-\lambda)F_\lambda^{-1}(z_j) &= u_j-\lambda F(u_j)-(1-\lambda)u_j \\ &= -\lambda(F(u_j)-u_j) \to 0 \quad \text{strongly.} \end{aligned}$$

The inequality $(*)$ permits us to complete the proof precisely as in Kirk [4] by showing that $\{z_j\}$ is necessarily a Cauchy sequence. (Replace F with $(1-\lambda)F_\lambda^{-1}$ and $\{u_j\}$ with $\{z_j\}$ in [4]. Note the misprint in [4, p. 411]; one should have

$$C_\xi=\bigcup_{k=1}^{\infty}\left(\bigcap_{i=k}^{\infty} S(u_j;\rho_0+\xi)\right).\Big)$$

It follows that $\{u_i\}$ is a Cauchy sequence:

$$\begin{aligned} \|u_i-u_j\| &\leqq \|u_i-\lambda F(u_i)-(u_j-\lambda F(u_j))\| + \|\lambda F(u_i)-\lambda F(u_j)\| \\ &\leqq \|z_i-z_j\| + \lambda_0\|u_i-u_j\|; \end{aligned}$$

hence $(1-\lambda_0)\|u_i-u_j\| \leqq \|z_i-z_j\| \to 0$ as $i, j \to \infty$. Therefore $\{u_j\}$ converges strongly, and this completes the proof.

REMARK. We make one final observation. For $A \subset X$, let $\gamma(A)$ denote the measure of noncompactness of A (cf. [5, p. 318]); $\gamma(A)=\inf\{d>0 \mid A$ can be covered by a finite number of sets of diameter less than or equal $d\}$. A continuous mapping $T: D \to X$ is said to be a *k-set contraction*, $k \geqq 0$, if for each bounded set A in D, $\gamma(T(A)) \leqq k\gamma(A)$. Now suppose $U: D \to X$ satisfies

$$(**) \qquad \gamma(A) \leqq \gamma(((1 + r)I - rU)(A)), \qquad A \subset D, r > 0.$$

For S, T, H as defined in the proof of Theorem 1, one can easily show that if $A \subset \bar{G}$ then $\gamma(S(A)) \leqq \gamma(T(A))$, and thus $\gamma(H(A)) \leqq \gamma(A)$. This proves that H is a 1-set contraction, and application of Petryshyn's theorem [6] in its full generality to H yields the fact that Theorem 1 holds for a more general class of mappings, namely the lipschitzian mappings which satisfy (**).

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