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ON THE COEFFICIENTS OF FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

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ABSTRACT. Let V_k be the class of normalised functions of bounded boundary rotation. For $f \in V_k$ define

$$M(r,f) = \max_{|z|=r} |f(z)|,$$

and let $L(r,f)$ denote the length of $f(|z|=r)$. Then if $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, it is shown that (i) $2M(r,f)<L(r,f)\leqq k\pi M(r,f)$, and (ii) $n^2|a_n|\leqq(3k/r^{n-1})M(r,f')$, $n\geqq 2$. The class Λ_k of meromorphic functions of boundary rotation is also studied and estimates for the coefficients are given.

1. **Introduction.** For fixed $k\geqq 2$, let V_k denote the class of normalised functions of boundary rotation at most $k\pi$; that is, $f\in V_k$ if and only if f is analytic in the open unit disc γ, $f'(z)\neq 0$ for $z\in\gamma$, $f(0)=0$, $f'(0)=1$, and f maps γ onto a domain with boundary rotation at most $k\pi$. Since the boundary rotation is the total variation of the argument of the boundary tangent vector (whenever such a tangent vector exists), we have (see e.g. [5]), with $z=re^{i\theta}$,

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{(zf'(z))'}{f'(z)} \right| d\theta \leqq k\pi. \tag{1}$$

V_2 is the class of normalised convex functions, and it is well known that for $2\leqq k\leqq 4$, V_k contains only univalent functions.

Suppose $f\in V_k$ and has Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

Then the problem $A_n(k)=\max|a_n|$ has been extensively studied, but remains largely unsolved. It is known that, for $k\geqq 2$,

$$A_2(k) = k/2, \quad A_3(k) = (k^2+2)/6, \quad A_4(k) = (k^3+8k)/24, \tag{3}$$

and, for $n\geqq 2$, that

$$|a_n| \leqq c(k)n^{k/2-1}, \tag{4}$$

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where $c(k)$ is a constant depending only upon k. (4) was given in [4] with $c(k)\to\infty$ as $k\to\infty$ and in [8] with $c(k)\to 0$ as $k\to\infty$. The function $f_0 \in V_k$ defined for $z\in\gamma$ by

$$f_0(z) = \frac{1}{\varepsilon k}\left[\left(\frac{1+\varepsilon z}{1-\varepsilon z}\right)^{k/2} - 1\right], \qquad |\varepsilon| = 1,$$

shows that equality may occur in each of (3), and also that the index of n in (4) is best possible.

A class of functions closely related to V_k is the class Λ_k $(k\geqq 2)$ of meromorphic functions defined as follows. Thc function g, given by

$$g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n, \tag{5}$$

belongs to Λ_k if and only if g is analytic in $\gamma' = \gamma\backslash\{0\}$, $g'(z)\neq 0$ for $z\in\gamma'$ and g maps γ' onto a domain with boundary rotation at most $k\pi$. For $g\in\Lambda_k$, we have [7], with $z=re^{i\theta}$,

$$\int_0^{2\pi}\left|\operatorname{Re}\frac{(zg'(z))'}{g'(z)}\right| d\theta \leqq k\pi. \tag{6}$$

If $g\in\Lambda_k$ is given by (5), then the problem $B_n(k)=\max|b_n|$ was considered in [6]. It was shown that

$$B_1(k) = k/2 \quad\text{and}\quad B_2(k) = k/6.$$

It was also shown in [6] that, for $n\geqq 1$, $|b_n|\leqq C(k)n^{k/2-3}$, for $k=2$ and $k\geqq 4$, where $C(k)\to\infty$ as $k\to\infty$.

Let $M(r,f)=\max_{|z|=r}|f(z)|$ and let $0<r<1$. The main purpose of this paper is to give estimates for the coefficients a_n and b_n in (2) and (5) in terms of $M(r,f')$ and $M(r,g')$ respectively. We shall also give an extremely simple proof of (4) with an improved constant $c(k)$. The methods of the paper show also that, *for all* $k\geqq 2$ and $n\geqq 2$, $|b_n|\leqq C(k)n^{k/2-3}$ where $C(k)\to 0$ as $k\to\infty$. This result improves the theorem given in [6], since the estimate is now valid for all $k\geqq 2$, and since $C(k)\to 0$ as $k\to\infty$.

2. Statement of results.

For V_k we have:

THEOREM 1. *Let $f\in V_k$ and be given by* (2). *Then, for any* $0<r<1$,

(i) $2M(r,f)<L(r)\leqq k\pi M(r,f)$ *where $L(r)$ is the image of $\gamma_r=\{z:|z|=r\}$ under f, and*

(ii) $n^2|a_n|\leqq(3k/r^{n-1})M(r,f')$, $n\geqq 2$.

REMARK. 1. A geometrical proof of (i) was given in [2] with a worse constant.

2. An example in [2] shows that the constant $k\pi$ in (i) is the best possible.

With the aid of this theorem we are able to prove the following corollaries:

COROLLARY 1. *Let $f\in V_k$ and be given by* (2). *Then, for $n\geqq 5$,*

$$n\,|a_n|\leqq e^4/2[(n/2)^{k/2}-1].$$

For $n=2, 3, 4$ the inequalities (3) *are better.*

COROLLARY 2. *Let $f\in V_k$ and be given by* (2). *Then, for $n\geqq 2$,*

$$n\,|a_n|\leqq 2ke^2(A(1-1/n)/\pi)^{1/2},$$

where $A(r)$ is the area of $f(\gamma_r)$.

This last result was obtained in [1], with essentially the same constants. In [2] it was also shown that for a bounded function in V_k, $na_n=o(1)$ as $n\to\infty$. The following extends this result.

COROLLARY 3. *Let $f\in V_k$ and be given by* (2). *Then if the area of $f(\gamma)$ is finite, $na_n=o(1)$ as $n\to\infty$, and the index of n is best possible.*

For Λ_k we have

THEOREM 2. *Let $g\in\Lambda_k$ and be given by* (5). *Then, for any $0<r<1$ and any $n\geqq 1$,*

$$n^2\,|b_n|\leqq (4k/r^{n-1})M(r,g').$$

We also have

COROLLARY 4. *Let $g\in\Lambda_k$ and be given by* (5). *Then, for $n\geqq 2$,*

$$|b_n|\leqq (32e^4k/2^{k/2})n^{k/2-3}.$$

The function $g_0\in\Lambda_k$, defined by

$$g_0'(z)=-\frac{1}{z^2}\frac{(1+z^2-2z(k-2)/(k+2))^{(k+2)/4}}{(1-z)^{(k-2)/2}}$$

shows that the index of n is best possible.

3. **Proofs of theorems.**

PROOF OF THEOREM 1. (i) Write

$$L(r)=\int_0^{2\pi}|zf'(z)|\,d\theta=\int_0^{2\pi}zf'(z)\exp[-i\arg zf'(z)]\,d\theta.$$

Then integration by parts gives

$$L(r) = \int_0^{2\pi} f(z)\exp[-i \arg zf'(z)]\, \partial_\theta(\arg zf'(z)) \leqq k\pi M(r,f)$$

on using (1). The left hand inequality is trivial.

(ii) We shall use the method of Clunie and Pommerenke [3], and shall need the following lemma.

LEMMA 1. *Let $f \in V_k$ and, for $n \geqq 1$, $z=re^{i\theta}$, put*

$$J_n(r) = \frac{1}{2\pi}\int_0^{2\pi} (zf'(z))'z^{n+1} \exp[-2i(\theta + \arg f'(z))]\, d\theta.$$

Then $|J_n(r)| \leqq 2kr^{n+1}M(r,f')$.

PROOF. Let $F(z)=(zf'(z))'/f'(z)$. Then $(\partial/\partial\theta)(\theta+\arg f'(z))=\operatorname{Re} F(z)$. Integration by parts shows that

$$J_n(r) = \frac{1}{\pi}\int_0^{2\pi} f_n(z)\exp[-2i(\theta + \arg f'(z))] \operatorname{Re} F(z)\, d\theta \tag{7}$$

where

$$f_n(z) = \int_0^z \zeta^n(\zeta f'(\zeta))'\, d\zeta.$$

Again using integration by parts we have

$$f_n(z) = z^{n+1}f'(z) - n\int_0^z \zeta^n f'(\zeta)\, d\zeta,$$

and so

$$|f_n(z)| \leqq r^{n+1}M(r,f') + nM(r,f')\int_0^r t^n\, dt \leqq 2r^{n+1}M(r,f').$$

From (7) we now have

$$|J_n(r)| \leqq \frac{2r^{n+1}}{\pi} M(r,f')\int_0^{2\pi} |\operatorname{Re} F(z)|\, d\theta,$$

and Lemma 1 now follows on using (1).

We now prove (ii). With $(zf'(z))'=f'(z)F(z)$ we write $F(z)=2\operatorname{Re} F(z)-F(z)^*$, where * denotes complex conjugate. Then with $z=re^{i\theta}$, we have

$$\begin{aligned} n^2a_n &= \frac{1}{2\pi r^{n-1}}\int_0^{2\pi} (zf'(z))' \exp[-i(n-1)\theta]\, d\theta \\ &= \frac{1}{\pi r^{n-1}}\int_0^{2\pi} f'(z)\operatorname{Re} F(z)\exp[-i(n-1)\theta]\, d\theta \\ &\quad - \frac{1}{2\pi r^{n-1}}\int_0^{2\pi} f'(z)F(z)^* \exp[-i(n-1)\theta]\, d\theta. \end{aligned}$$

Hence

$$\begin{aligned} n^2 |a_n| &\leqq \frac{1}{\pi r^{n-1}} \int_0^{2\pi} |f'(z)|\, |\mathrm{Re}\, F(z)|\, d\theta \\ &\quad + \frac{1}{2\pi r^{n-1}} \left| \int_0^{2\pi} f'(z)^* F(z) \exp[i(n-1)\theta]\, d\theta \right| \\ &= P_1 + P_2, \quad \text{say,} \end{aligned}$$

where in P_2 we have taken the complex conjugate.

From (1) we obtain at once $P_1 \leqq (k/r^{n-1})M(r,f')$. Now

$$f'(z)^* F(z) = (zf'(z))' \exp[-2i \arg f'(z)],$$

and so

$$\begin{aligned} P_2 &= \frac{1}{2\pi r^{n-1}} \left| \int_0^{2\pi} (zf'(z))' \exp[i(n+1)\theta] \exp[-2i(\theta + \arg f'(z))]\, d\theta \right| \\ &= (1/r^{2n})\, |J_n(r)|. \end{aligned}$$

Thus from Lemma 1, $P_2 \leqq (2k/r^{n-1})M(r,f')$. Hence

$$n^2 |a_n| \leqq (3k/r^{n-1})M(r,f'),$$

which proves (ii).

Proof of Corollary 1. It is well known [5] that for $f \in V_k$,

$$|f(re^{i\theta})| \leqq \frac{1}{k}\left[\left(\frac{1+r}{1-r}\right)^{k/2} - 1\right].$$

With $n \geqq 5$, choose $r = 1 - 4/n$, then from Theorem 1 (i), with $z = re^{i\theta}$,

$$n\, |a_n| \leqq \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)|\, d\theta \leqq \frac{1}{2r^n}\left[\left(\frac{1+r}{1-r}\right)^{k/2} - 1\right] < \frac{e^4}{2}\left[\left(\frac{n}{2}\right)^{k/2} - 1\right].$$

Proof of Corollary 2. Note that

$$\begin{aligned} rM(r,f') &\leqq \sum_{n=1}^{\infty} n\, |a_n|\, r^n \\ &\leqq \left(\sum_{n=1}^{\infty} n\, |a_n|^2\, r^n\right)^{1/2} \left(\sum_{n=1}^{\infty} nr^n\right)^{1/2} \leqq \left(\frac{A(\sqrt{r})}{\pi}\right)^{1/2} \Big/ (1-r). \end{aligned}$$

For $n \geqq 2$, choose $r = (1 - 1/n)^2$, and the result follows at once from Theorem 1 (ii).

Proof of Corollary 3. This follows at once from Theorem 1 (ii) on noting that if the area of $f(\gamma)$ is finite, then $M(r,f') = o(1)/(1-r)$ as $r \to 1$.

The function $f_\alpha : f_\alpha(z) = (1/\alpha)[1 - (1-z)^\alpha]$ for $0 < \alpha < 1$ is convex and bounded and shows that the index of n is best possible.

Proof of Theorem 2. We need a lemma analogous to Lemma 1.

LEMMA 2. *Let $g \in \Lambda_k$ and, for $n \geqq 2$, $z = re^{i\theta}$, put*

$$K_n(r) = \frac{1}{2\pi} \int_0^{2\pi} (zg'(z))' z^{n+1} \exp[-2i(\theta + \arg g'(z))] \, d\theta.$$

Then $|K_n(r)| \leqq 3kr^{n+1} M(r, g')$.

PROOF. Let $G(z) = (zg'(z))'/g'(z)$. Then G is analytic in γ' and $(\partial/\partial\theta)(\theta + \arg g'(z)) = \operatorname{Re} G(z)$. Integration by parts shows that

$$K_n(r) = \frac{1}{\pi} \int_0^{2\pi} g_n(z) \exp[-2i(\theta + \arg g'(z))] \operatorname{Re} G(z) \, d\theta$$

where $g_n(z) = \int_0^z \zeta^n (\zeta g'(\zeta))' \, d\zeta$ (Note: $n \geqq 2$ assures regularity of the integrand at $\zeta = 0$.)

Again using integration by parts,

$$g_n(z) = z^{n+1} g'(z) - n \int_0^z \zeta^n g'(\zeta) \, d\zeta,$$

and so $|g_n(z)| \leqq 3r^{n+1} M(r, g')$. Exactly as in Lemma 1 we now find that $|K_n(r)| \leqq 3kr^{n+1} M(r, g')$, which proves Lemma 2.

We now prove Theorem 2. For $n \geqq 2$ we have

$$\begin{aligned} n^2 b_n &= \frac{1}{2\pi r^{n-1}} \int_0^{2\pi} (zg'(z))' \exp[-i(n-1)\theta] \, d\theta \\ &= \frac{1}{\pi r^{n-1}} \int_0^{2\pi} g'(z) \exp[-i(n-1)\theta] \operatorname{Re} G(z) \, d\theta \\ &\quad - \frac{1}{2\pi r^{n-1}} \int_0^{2\pi} g'(z) G(z)^* \exp[-i(n-1)\theta] \, d\theta. \end{aligned}$$

Hence

$$\begin{aligned} n^2 |b_n| &\leqq \frac{1}{\pi r^{n-1}} \int_0^{2\pi} |g'(z)| \, |\operatorname{Re} G(z)| \, d\theta \\ &\quad + \frac{1}{2\pi r^{n-1}} \left| \int_0^{2\pi} g'(z)^* G(z) \exp[i(n-1)\theta] \, d\theta \right| \\ &= Q_1 + Q_2, \quad \text{say.} \end{aligned}$$

As before, $Q_1 \leqq (k/r^{n-1}) M(r, g')$. Also,

$$\begin{aligned} Q_2 &= \frac{1}{2\pi r^{n-1}} \left| \int_0^{2\pi} (zg'(z))' \exp[i(n+1)\theta] \exp[-2i(\theta + \arg g'(z))] \, d\theta \right| \\ &= (1/r^{2n}) \, |K_n(r)| \leqq (3k/r^{n-1}) M(r, g') \end{aligned}$$

by Lemma 2. Thus, for $n \geqq 2$,

$$n^2 |b_n| \leqq (4k/r^{n-1}) M(r, g').$$

An elementary estimate using the Cauchy integral formula shows the above estimate is also true for $n=1$, and so the proof is complete.

PROOF OF COROLLARY 4. In **[6]** it is shown that $g \in \Lambda_k$ if and only if there exists $f \in V_k$ with $a_2=0$ such that

$$-z^2 g'(z) = 1/f'(z), \tag{8}$$

and that

$$M(r, g') \leqq \frac{(1+r)^{k/2+1}}{r^2(1-r)^{k/2-1}}. \tag{9}$$

We remark that (9) is certainly not sharp, but is sufficient for our purpose. Let $n \geqq 5$ and choose $r=1-4/n$. Then from Theorem 2 and (9) we have

$$|b_n| \leqq (32ke^4/2^{k/2}) n^{k/2-3}. \tag{10}$$

It remains only to show that this estimate is valid for $n=2, 3, 4$. For $n=2$, (10) follows since $|b_2| \leqq k/6$. Using (8) together with the condition $a_2=0$, it is easily seen on equating coefficients that $|b_3| \leqq k^2/24+k/12$, which gives (10) for $n=3$. Similarly one can obtain $|b_4| \leqq k^2/24+k/20$, which again gives (10) for $n=4$.

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