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A CONDITION FOR ANALYTIC STRUCTURE[1]

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ABSTRACT. Let X be a compact Hausdorff space, A a uniform algebra on X, M the maximal ideal space of A. Let $f \in A$ and let W be a component of $C \setminus f(X)$. Suppose that, for all $\lambda \in W$, $f^{-1}(\lambda) = \{x \in M \mid f(x) = \lambda\}$ is at most countable. Then there is an open dense subset U of $f^{-1}(W)$ which can be given the structure of a one-dimensional complex analytic manifold so that for all $g \in A$, g is analytic on U.

Let X be a compact Hausdorff space, let A be a uniform algebra on X, and let M be the maximal ideal space of A.

THEOREM. *Let $f \in A$ and let W be a component of $C \setminus f(X)$. Suppose that for all $\lambda \in W$, $f^{-1}(\lambda) = \{x \in M \mid f(x) = \lambda\}$ is at most countable. Then there is an open dense subset of $f^{-1}(W)$ which can be given the structure of a one-dimensional complex analytic manifold so that the functions in A become analytic there.*

This theorem partially generalizes results for the case when $f^{-1}(\lambda)$ is finite for all λ in a sufficiently large subset of W, which are essentially contained in a paper of E. Bishop ([**1**], see Theorem 11.2 in [**2**]). A key result in the proof of this earlier result is also important in the proof of our theorem (see Lemma 13 in [**1**] or Theorem 10.7 in [**2**]).

DEFINITION. Let $p \in M$, and let Φ be a continuous one-to-one map from $\{|z| < 1\}$ into M, with $\Phi(0) = p$. The set $\{\Phi(z) \mid |z| < 1\}$ is called an analytic disk through p if for all $h \in A$, $h \circ \Phi$ is analytic on $\{|z| < 1\}$.

LEMMA 1. *Let $f \in A$ and suppose that*:

(a) $|f| = 1$ *on* X;

(b) $\exists\, p \in M$ *with* $f(p) = 0$;

(c) $\exists$ *a closed subset* Γ_0 *of* $\{|z| = 1\}$ *having positive linear measure such that for each* $\lambda \in \Gamma_0$ *there is a unique point* $q \in X$ *with* $f(q) = \lambda$.

Then $f^{-1}(\{|z| < 1\})$ *is an analytic disk through* p.

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We also need the following elementary result (Lemma 11.1 in [2]).

LEMMA 2. *Let $f \in A$ and let W be a component of $\boldsymbol{C} \setminus f(X)$. Fix $\lambda \in W$. If f takes on the value λ on M, then f takes on every value in W on M.*

One further result is needed.

LEMMA 3. *Let $f \in A$ and let W be a component of $\boldsymbol{C} \setminus f(X)$. Let $z \in W$, and suppose that $f^{-1}(z)$ is at most countable. Given a neighborhood $\mathcal{O}$ of a point $x \in f^{-1}(z)$, there is a compact neighborhood N of x with the properties that:*

(a) *N is A-convex, i.e., the maximal ideal space of $\bar{A}|_N$ is N;*

(b) *$z \notin f(\partial N)$;*

(c) *$N \subseteq \mathcal{O} \cap f^{-1}(W)$.*

Furthermore, $f(N)$ is a neighborhood of z in $\boldsymbol{C}$.

PROOF OF LEMMA 3. Choose $\varepsilon > 0$ and functions $g_1, g_2, \cdots, g_n \in A$ so that

$$N = \{y \in M \mid |g_j(y) - g_j(x)| \leqq \varepsilon,\ 1 \leqq j \leqq n\} \subseteq f^{-1}(W) \cap \mathcal{O}.$$

Since $f^{-1}(z)$ is at most countable, we may adjust ε so that, for all $y \in f^{-1}(z)$, $|g_j(y) - g_j(x)| \neq \varepsilon$, $j = 1, 2, \cdots, n$. Now if $y \in \partial N$, then $|g_j(y) - g_j(x)| = \varepsilon$ for some j, so $f^{-1}(z) \cap \partial N = \varnothing$ and it is clear that (a), (b) and (c) are satisfied.

By the local maximum modulus principle, the Shilov boundary of $\bar{A}|_N$ is contained in ∂N (note that N does not meet the Shilov boundary of A). Hence $\bar{A}|_N$ may be regarded as a uniform algebra on ∂N. Also, $z \in f(N) \setminus f(\partial N)$. By Lemma 2, $f(N)$ contains the component of z in $\boldsymbol{C} \setminus f(\partial N)$, so $f(N)$ is a neighborhood of z as claimed.

PROOF OF THEOREM. We will prove the following.

Assertion. If $f^{-1}(W)$ is nonempty, $\exists\, p \in f^{-1}(W)$ with a neighborhood in M which is an analytic disk through p.

Assume for the moment that the assertion has been established. Given any $x \in f^{-1}(W)$ and any neighborhood $\mathcal{O}$ of x, we may apply Lemma 3 to find a compact neighborhood N of x with properties (a), (b) and (c). Then the assertion may be applied to $f|_N \in \bar{A}|_N$, to yield a point $p \in \mathcal{O}$ with a neighborhood which is an analytic disk through p. Thus the set of such points is dense in $f^{-1}(W)$, and it is obviously open, so the Theorem follows.

PROOF OF ASSERTION. Suppose, on the contrary, that no point of $f^{-1}(W)$ has a neighborhood which is an analytic disk. We will use this assumption to find a point $z \in W$ such that $f^{-1}(z)$ is uncountable, contradicting the hypothesis of the Theorem. The uncountable set will consist of limit points obtained from sequences $\{x_{i_1 i_2 \cdots i_n}\}_{n=1}^{\infty}$ in M, which we will now define inductively.

Step 1. Since $f^{-1}(W)$ is nonempty, by Lemma 2 we have $f(f^{-1}(W))=W$, so by Lemma 1 and the assumption that no point of $f^{-1}(W)$ has a neighborhood which is an analytic disk, there are distinct points $x_0, x_1 \in M$ and a corresponding $z_1 \in W$ such that $f(x_0)=f(x_1)=z_1$. By Lemma 3 there are disjoint compact A-convex neighborhoods N_{i_1} of x_{i_1} with the properties $N_{i_1} \subseteq f^{-1}(W)$, $z_1 \notin f(\partial N_{i_1})$, and $f(N_{i_1})$ is a neighborhood of z_1, $i_1=0, 1$. Choose ε_1 with $0<\varepsilon_1<1$ so that

$$(\mathrm{A}_1) \qquad \{|z - z_1| \leqq \varepsilon_1\} \subseteq f(N_{i_1}) \backslash f(\partial N_{i_1}), \qquad i_1 = 0, 1.$$

Fix i_1 and look at

$$M_{i_1} = \{y \in N_{i_1} \mid |f(y) - z_1| \leqq \varepsilon_1\} = N_{i_1} \cap f^{-1}(\{|z - z_1| \leqq \varepsilon_1\}).$$

M_{i_1} is a nonempty compact A-convex set, so the maximal ideal space of $\bar{A}|_{M_{i_1}}$ is M_{i_1}. Let $X_{i_1}=\{y \in N_{i_1} \mid |f(y)-z_1|=\varepsilon_1\}$.

Observe that $\partial M_{i_1} \subseteq (\partial N_{i_1} \cup X_{i_1}) \cap M_{i_1} = X_{i_1}$ since $\partial N_{i_1} \cap M_{i_1} = \varnothing$ by (A_1). Also observe that M_{i_1} does not meet the Shilov boundary of A. By the local maximum modulus principle, the Shilov boundary of $\bar{A}|_{M_{i_1}}$ is contained in X_{i_1}. Thus we may apply Lemma 1 with $X=X_{i_1}$, $A=\bar{A}|_{X_{i_1}}$, $M=M_{i_1}$, $f=(f-z_1)/\varepsilon_1$, to conclude that (since there is no analytic disk through any point of $f^{-1}(W)$):

$$(\mathrm{B}_1) \qquad \text{linear measure } \{z \in \boldsymbol{C} \mid |z - z_1| = \varepsilon_1 \text{ and } f^{-1}(z) \text{ is unique in } N_{i_1}\} = 0.$$

(Note that $f^{-1}(z) \cap N_{i_1} = f^{-1}(z) \cap M_{i_1}$ if $|z-z_1|=\varepsilon_1$. The set described in statement (B_1) is measurable since Lemma 3 implies that it is closed. In fact, suppose that $\zeta_1, \zeta_2, \cdots \in \boldsymbol{C}$, $\zeta_j \to \zeta \in \boldsymbol{C}$, $|\zeta_j - z_1| = \varepsilon_1$ and there is precisely one $\alpha_j \in N_{i_1}$ such that $f(\alpha_j)=\zeta_j$, $j=1, 2, \cdots$. Let $\alpha \in f^{-1}(\zeta) \cap N_{i_1}$ and let V be any neighborhood of α in M. By Lemma 3, $f(V \cap N_{i_1})$ is a neighborhood of ζ in $\boldsymbol{C}$, hence ζ_j is in $f(V \cap N_{i_1})$ for all $j \geqq j_0$ for some j_0. But $f^{-1}(\zeta_j) \cap N_{i_1} = \{\alpha_j\}$, so α_j is in V for $j \geqq j_0$. Thus α_j converges to α, and α is unique.)

This completes the first step.

Inductive hypothesis. Suppose that $z_n \in W$, $x_a \in M$, $N_a \subseteq M$ (where $a=i_1 \cdots i_n$) and $\varepsilon_n > 0$ have been chosen for $i_j = 0, 1$, $1 \leqq j \leqq n$. Assume that they satisfy the conditions: the 2^n points x_a are all distinct; $f(x_a)=z_n$; N_a is a compact A-convex neighborhood of x_a; $N_a \cap N_b = \varnothing$ (where $b=j_1 \cdots j_n$) unless $i_1=j_1, \cdots, i_n=j_n$; $N_a \subseteq$ interior $N_{i_1 \cdots i_{n-1}}$, if $n>1$; $\varepsilon_n < 1/n^2$;

$$(\mathrm{A}_n) \qquad \{|z - z_n| \leqq \varepsilon_n\} \subseteq f(N_a) \backslash f(\partial N_a);$$

$$(\mathrm{B}_n) \qquad \text{linear measure } \{z \in C \mid |z - z_n| = \varepsilon_n \text{ and } f^{-1}(z) \text{ is unique in } N_a\} = 0.$$

Step $n+1$. By (B_n), $\exists\, z_{n+1} \in \{|z-z_n|=\varepsilon_n\}$ such that for all $i_1 \cdots i_n$, there are two distinct points $x_c \in N_a$ (where $c=i_1 \cdots i_{n+1}$, $i_{n+1}=0, 1$) with $f(x_c)=z_{n+1}$. By (A_n), we have $x_c \in N_a \backslash f^{-1}(f(\partial N_a))$. Fix $a=i_1 \cdots i_n$ and choose disjoint open sets $\mathcal{O}_0$, $\mathcal{O}_1$ with

$$x_c \in \mathcal{O}_{i_{n+1}} \subseteq N_a \backslash f^{-1}(f(\partial N_a)), \qquad i_{n+1} = 0, 1.$$

Fix $i_{n+1}=0$ or 1 and apply Lemma 3 with

$$A = \bar{A}|_{\partial N_a}, \quad X = \partial N_a, \quad M = N_a, \quad f = f|_{N_a},$$
$$W = \text{component of } z_{n+1} \text{ in } \boldsymbol{C} \backslash f(\partial N_a), \quad z = z_{n+1}, \quad \mathcal{O} = \mathcal{O}_{i_{n+1}}, \quad x = x_c.$$

We conclude that there exist disjoint compact A-convex neighborhoods N_c of x_c, $i_{n+1}=0, 1$, with the properties $z_{n+1} \notin f(\partial N_c)$,

$$N_c \subseteq N_a \backslash f^{-1}(f(\partial N_a)),$$

$f(N_c)$ is a neighborhood of z_{n+1}. Do this for all indices $c=i_1 \cdots i_{n+1}$, $i_j=0, 1$, $1 \leqq j \leqq n+1$.

From the definition of the N_c it follows that we can choose an ε_{n+1} with $0<\varepsilon_{n+1}<1/(n+1)^2$ so that for all $i_1 \cdots i_{n+1}$ we have

$$(A_{n+1}) \qquad \{|z - z_{n+1}| \leqq \varepsilon_{n+1}\} \subseteq f(N_c) \backslash f(\partial N_c).$$

Fix $c=i_1 \cdots i_{n+1}$ and define

$$M_c = \{y \in N_c \mid |f(y) - z_{n+1}| \leqq \varepsilon_{n+1}\},$$
$$X_c = \{y \in N_c \mid |f(y) - z_{n+1}| = \varepsilon_{n+1}\}.$$

As in Step 1, the maximal ideal space $\bar{A}|_{M_c}$ is M_c and the Shilov boundary of $\bar{A}|_{M_c}$ is contained in X_c. ((A_{n+1}) guarantees that $\partial M_c \subseteq X_c$.) We may therefore apply Lemma 1 with

$$X = X_c, \quad A = \bar{A}|_{X_c}, \quad M = M_c, \quad f = (f - z_{n+1})/\varepsilon_{n+1}.$$

We conclude that

$$(B_{n+1}) \qquad \text{linear measure } \{z \in \boldsymbol{C} \mid |z - z_{n+1}| = \varepsilon_{n+1} \text{ and } f^{-1}(z) \text{ is unique in } N_c\} = 0.$$

This equation holds for all indices $c = i_1 \cdots i_{n+1}$, $i_j = 0, 1$, $1 \leqq j \leqq n+1$, and the induction is complete.

By the above construction $\{z_n\}$ is Cauchy, so $z_n \to z$ for some $z \in \boldsymbol{C}$; $z \in W$ since, for each n, $z_n \in f(N_a) \subseteq f(N_0) \cup f(N_1)$, a compact subset of W.

Let $I=(i_1, i_2, \cdots, i_n, \cdots)$ be an infinite sequence of 0's and 1's. Some subnet of the sequence $x_{i_1}, x_{i_1 i_2}, x_{i_1 i_2 i_3}, \cdots$ converges to a point $x_I \in M$, and by continuity $f(x_I)=z$. In this way we associate an x_I with each I.

Since there are uncountably many distinct I's, the proof will be complete if we show that $x_I \neq x_J$ whenever $I \neq J$. Suppose therefore that $i_1 = j_1$, $i_2 = j_2, \cdots, i_m = j_m, i_{m+1} \neq j_{m+1}$. We know that $x_{i_1} \in N_{i_1}$, $x_{i_1 i_2} \in N_{i_1 i_2}, \cdots$ and $N_{i_1} \supseteq N_{i_1 i_2} \supseteq N_{i_1 i_2 i_3} \cdots$ by the inductive hypotheses. Thus $x_I \in \bigcap_{n=1}^{\infty} N_a$. Similarly $x_J \in \bigcap_{n=1}^{\infty} N_b$ where $b = j_1 \cdots j_n$. But $N_{i_1 \cdots i_{m+1}} \cap N_{j_1 \cdots j_{m+1}} = \varnothing$ since $i_{m+1} \neq j_{m+1}$, so $x_I \neq x_J$.

B. Cole has pointed out that if one merely assumes that $f^{-1}(\lambda)$ is countable for λ in a subset of W of positive plane measure, the same proof shows that there is at least one point $p \in f^{-1}(W)$ with a neighborhood in M which is an analytic disk through p.

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