CIRCLE ACTIONS ON HOMOTOPY SPHERES
BOUNDING PLUMBING MANIFOLDS

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Abstract. Smooth circle actions are constructed on certain homotopy spheres not previously known to admit such actions.

In this paper we shall prove the following two results:

Proposition A. Let $\Sigma^8$ be any homotopy 8-sphere. Then there is a smooth semifree circle action on $\Sigma^8$ with $S^4$ as its fixed point set.

Proposition B. Let $\Sigma^{10}$ be any homotopy 10-sphere bounding a spin manifold. Then there is a smooth semifree circle action on $\Sigma^{10}$ with $S^4$ as its fixed point set.

Combining these with other results, we know that any smooth manifold $\Sigma^n$ which is piecewise-differentiably homeomorphic to $S^n$, bounds a spin manifold, and satisfies $n \leq 13$ has a smooth circle action. The above propositions imply the cases $n=8, 10$, while the cases $n=7, 11, 12$ follow because $\Gamma_n = \partial P_{n+1}$ in these cases and every homotopy sphere in $\partial P_{n+1}$ $(n \geq 5)$ has a semifree circle action with a homotopy $(n-4)$-sphere as its fixed point set (e.g., see [3]). Finally, the cases $n=9, 13$ follow from the above remark on $\partial P_{n+1}$ and results of Bredon [1].

Undoubtedly, the central difficulty in obtaining connected Lie group actions on homotopy spheres is the lack of a manageable construction for an arbitrary such manifold. The value of such a realization is obvious in the construction of large orthogonal group actions on homotopy spheres bounding $\pi$-manifolds. Bredon’s construction of smooth $S^1$ and $S^3$ actions on homotopy spheres in the image of the Milnor-Munkres-Novikov pairing [1] is another illustration of the usefulness of an explicit construction for a given homotopy sphere. In this paper we shall show that certain homotopy spheres in the image of the Milnor plumbing pairing $\sigma_{p,q}: \pi_q(SO_p) \times \pi_p(SO_q) \to \Gamma_{p+q+1}$ (see [4] or [5]) also have smooth circle actions.

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actions. Propositions A and B follow from this, results of D. Frank [4] stating that the image of $\sigma_{3,4}$ is $\Gamma_8$ and the image of $\sigma_{3,6}$ is $2\Gamma_{10}$, and some homotopy-theoretic computations. I am indebted to D. Frank for suggesting that results like Propositions A and B might be obtainable from [4].

1. Constructing semifree circle actions. We shall construct smooth semifree circle actions on certain homotopy spheres in the image of $\sigma_{p,q}$ by the familiar technique of pasting two $S^1$ manifolds together via an equivariant diffeomorphism. However, we shall be pasting along a semifree $S^1$ manifold rather than a free $S^1$ manifold (which is the usual way of studying semifree actions; e.g., see [2, Chapter III]).

Let $p$, $q$ and $r$ be positive integers satisfying $2r \leq q+1$. Then $S^1$ acts orthogonally on $R^{q+1}$ via the representation containing $r$ copies of the standard 2-dimensional representation and a trivial representation; call this representation $A_r$. Then the induced smooth action on $S^p \times S^q$ given by

\begin{equation}
(z \cdot (x, y)) = (x, A_r(z)y), \quad x \in S^p, \ y \in S^q
\end{equation}

is semifree, and its fixed point set is $S^p \times S^{q-2r}$.

The orthogonal action $A_r$ on $S^q$ may be readily interpreted in terms of the homeomorphism from $S^q$ to the join $S^{2r-1} \ast S^{q-2r}$; namely, it is the free linear action in the first join coordinate (an element of $S^{2r-1}$) and the trivial action on the other two join coordinates (elements of $S^{q-2s}$ and $[0, 1]$ respectively). One advantage of this interpretation is that the orbit space projection is immediately recognized as the join of the canonical map $p_{r-1}: S^{2r-1} \to CP^{r-1}$ with the identity on $S^{q-2r}$. It is well known that this map is homotopically equivalent to the $(q-2r+1)$-fold suspension of $P^{r-1}$.

Lemma 1.2. Let $\alpha \in \pi_p(SO_q)$ be in the image of $\pi_p(U_r)$ and let $\beta \in \pi_q(SO_p)$ be in the image of

$$(S^{q-2r+1}p_{r-1})^*: [S^{q-2r+1}CP^{r-1}, SO_p] \to \pi_q(SO_p).$$

Then the homotopy $(p+q+1)$-sphere corresponding to $\sigma_{p,q}(\alpha, \beta)$ has a semifree circle action whose fixed point set is an ordinary $(p+q+1-2r)$-sphere.

Proof. Let $\varphi: S^p \to U_r$ be a smooth map representing $\alpha$ and let $f$ be the diffeomorphism of $S^p \times S^q$ defined by $f(x, y) = (x, \varphi(x)y)$, where $U_r$ acts orthogonally on $S^q$; then $f$ is equivariant with respect to the $S^1$ action defined in (1.1). Let $\psi: S^q \to SO_p$ be a continuous map representing $\beta$; the assumption that $\beta$ factors through $S^{q-2r+1}CP^{r-1}$ implies that $\psi$ may be assumed to be constant on the orbits of the action $A_r$. If $SO_p$ is taken with
the trivial $S^1$ action, then the smooth equivariant approximation theorem [7, 1.12] implies that $\psi$ may also be assumed to be smooth. Hence the diffeomorphism $g(x, y) = (\psi(y)x, y)$ is also equivariant with respect to the $S^1$ action of (1.1); therefore, the commutator $[f, g]$ is also $S^1$ equivariant.

The action given in (1.1) obviously extends to smooth $S^1$ actions on the manifolds $D^{p+1} \times S^q$ and $S^p \times D^{q+1}$. Thus if these two manifolds with boundary are pasted together via the equivariant diffeomorphism $[f, g]$, the resulting closed manifold has a smooth circle action. But the resulting closed manifold is merely the homotopy sphere corresponding to $\sigma(\alpha, \beta)$; e.g., see [5, remark preceding 2.1]. Since the fixed point sets of the extended actions on $D^{p+1} \times S^q$ and $S^p \times D^{q+1}$ are $D^{p+1} \times S^{q-2r}$ and $S^p \times S^{q-2r}$ respectively, the fact that $[f, g]$ is the identity on $S^p \times S^{q-2r}$ implies that the fixed point set of the action on $\sigma(\alpha, \beta)$ is the ordinary sphere

$$S^{p+q+2r+1} = D^{p+1} \times S^{q-2r} \cup S^p \times D^{q+1}.$$ 

It follows immediately that the circle action constructed above is semifree.

2. Proof of main results. The proofs of Propositions A and B are roughly parallel, although the latter is somewhat more complicated.

Proof of Proposition A. Let $\alpha \in \pi_3(SO_4)$ map to the generator of $\pi_3(SO) = \mathbb{Z}$, let $\beta$ generate $\pi_3(SO_3) = \mathbb{Z}$, and let $\eta$ generate $\pi_4(S^3) = \mathbb{Z}_2$. According to [4, Example 1, p. 565], the element $\sigma_3(\alpha, \beta, \eta) \in \Gamma_8$ is nonzero for any choice of $\alpha$. On the other hand, it is well known that $\alpha$ may be chosen to lie in the image of $\pi_3(U_3)$. Since $\eta$ is the orbit space projection $S^0 * S^3 \to S^0 * S^2$ described in §1, Proposition A follows immediately from Lemma 1.2.

The proof of Proposition B depends on homotopy computations involving 3-primary components; it will be convenient to localize all spaces at the prime 3. A discussion of localization functors appears in [6, Chapter II]; the main property we need is that the topological localization map $I_X: X \to X_{(p)}$ is given in homotopy by the algebraic localization $\pi_*(X) \to \pi_*(X) \otimes \mathbb{Z}(p)$. The following generalization is well known.

Lemma 2.1. Let $X$ be a finite CW complex and let $Y$ be a topological group, so that $[SX, Y]$ is naturally an abelian group. Then there is a natural isomorphism $\alpha_X: [SX, Y_{(p)}] \to [SX, Y] \otimes \mathbb{Z}(p)$ such that $\alpha_X|_X$ is the canonical map $[SX, Y] \to [SX, Y] \otimes \mathbb{Z}(p)$.

We shall need the following computational result:

Lemma 2.2. Let $\nu'$ generate $\pi_6(S^3_{(3)}) = \mathbb{Z}_3$, and let $g: S^3 \to \mathbb{C}P^2$ be the canonical projection. Then $\nu'$ is in the image of the map $(Sg)^*: [SCP^2, S^3_{(3)}] \to \pi_6(S^3_{(3)})$. 

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Proof. Recall that the first two nonzero homotopy groups of \( S^3 \) are \( \mathbb{Z}(3) \) in the dimension 3 and \( \mathbb{Z}_3 \) in dimension 6. These are related by the nontrivial \( k \)-invariant \( P^1 \in H^7(\mathbb{Z}(3), 3; \mathbb{Z}_3) = \mathbb{Z}_3 \).

By the exactness of the Puppe sequence and the mapping cone sequence
\[
S^6 \xrightarrow{g} CP^2 \xrightarrow{h} CP^3 \to S^6,
\]
it suffices to show that \( h^*: \pi_6(S^3) \to [CP^3, S^3] \) is the trivial map. But this is a straightforward consequence of the following two facts:

(i) \( P^1 \) is the first \( k \)-invariant of \( S^3_3 \).
(ii) The suspension of \( P^1 \) (i.e., \( i^3 \in H^k(\mathbb{Z}(3), 2) \)) has a nontrivial image in \( H^6(CP^3; \mathbb{Z}_3) \).

Remark. Since \( S^3 \) is the universal covering group of \( SO_3 \) and \( \pi_1(SO_3) = \mathbb{Z}_2 \), there is an obvious homotopy equivalence from \( S^3_3 \) to \( S^3_3(3) \); consequently, the lemma remains true if \( S^3 \) is replaced by \( SO_3 \).

Proof of Proposition B. Let \( \alpha \in \pi_3(SO_3) = \mathbb{Z} \) be a generator, and let \( \beta \) generate the 3-primary component of \( \pi_6(SO_3) = \mathbb{Z}_{12} \). As in the proof of Proposition A, we know that \( \alpha \) is in the image of \( \pi_3(U_3) \). On the other hand, Lemma 2.2 implies that \( \beta \) is in the image of \( (Sg)^*: [SCP^2, SO_3] \to \pi_6(SO_3) \). Since \( \sigma_3, 6(\alpha, \beta) \) is nonzero by results of Frank [4, Example 2, p. 565], some exotic 10-sphere bounding a Spin manifold has a circle action of the desired type by Lemma 1.2. Since \( bSpin_{11} = \mathbb{Z}_3 \) is cyclic, an action on the other one may be constructed by taking an equivariant connected sum of this action with itself along the fixed point set.

References

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