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ON THE LATTICE OF SUBALGEBRAS OF A BOOLEAN ALGEBRA

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ABSTRACT. The lattice of all subalgebras of a Boolean algebra is characterized.

1. **Introduction.** The following three facts are known: The lattice $S(\mathfrak{A})$ of all subalgebras of an algebra $\mathfrak{A}$ can be characterized as an algebraic lattice (G. Birkhoff and O. Frink, Jr. [2]), and the finitely generated subalgebras are characterized in this lattice as compact elements; if $\mathfrak{A}$ is the 2^n-element Boolean algebra, then $S(\mathfrak{A})$ is dually isomorphic to the partition lattice on n objects (G. Birkhoff [1]); a finitely generated Boolean algebra is finite. Combining these three facts we can conclude that the lattice $L=S(B)$ of all subalgebras of a Boolean algebra has the following property: (P) L is an algebraic lattice and for each compact element a of L, $(a]$ is dually isomorphic to a finite partition lattice.

Let L be a lattice having this property (P). Then, with each compact element a of L we can associate a finite Boolean algebra $B(a)$ such that $S(B(a))\cong(a]$. Thus, if a and b are compact elements of L with $a\leqq b$, there is a natural embedding φ_{ab} of $S(B(a))$ into $S(B(b))$ such that $S(B(a))\varphi_{ab}$ is a principal ideal of $S(B(b))$. We are going to show that there is a unique embedding ψ_{ab} of $B(a)$ into $B(b)$ which induces the embedding φ_{ab}. The $B(a)$ with the ψ_{ab} form a direct family of Boolean algebras. We prove that if B is the direct limit of this family, then $S(B)$ is isomorphic to L.

This result has two corollaries. Firstly, property (P) characterizes the lattice of all subalgebras of a Boolean algebra; secondly, $S(B)$ determines B up to isomorphism.

The second of these results is due to D. Sachs [6] and N. D. Filippov [3]. The first author also gave a characterization of $S(B)$ as a subsystem of an infinite partition lattice.

We refer to [4] and [5] for the undefined concepts, notations, and basic results.

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2. **Basic lemmas.** To start with, we develop a series of lemmas on which our proof of the main result is based. Our main purpose is to obtain some information on the relationship between two finite Boolean algebras A and B provided $S(A)$ and $S(B)$ are suitably related.

LEMMA 1. *Let A be a finite Boolean algebra. Let $p \in A$, $p \neq 0, 1$. Then p or p' is an atom of A if and only if $|[p] \vee [x]| \leqq 8$, for each $x \in A$.*

PROOF. p or p' is an atom of A if and only if for every x, one of $x \wedge p$, $x' \wedge p$, $x \wedge p'$, $x' \wedge p'$ is 0 which, in turn, is equivalent to the fact that the number of atoms of $[p] \vee [x]$, which are among the elements named, is at most three, that is, $|[p] \vee [x]| \leqq 8$.

By means of this lemma, the following result can be established.

LEMMA 2. *Let A and B be finite Boolean algebras. Let $\varphi: S(A) \rightarrow S(B)$ be a one-to-one homomorphism such that $S(A)\varphi$ is a principal ideal of $S(B)$. Then, for each atom x of A, there is an atom $\bar{x}$ of $A\varphi$ such that $[x]\varphi = [\bar{x}]$ in $A\varphi$.*

PROOF. Let x be an atom of A. Since $S(A)\varphi$ is a principal ideal of $S(B)$, there exists $b \in A\varphi$, $b \neq 0, 1$, of $A\varphi$ such that $[x]\varphi = [b]$. Let t be an arbitrary element of $A\varphi$. Then $[t] = [k]\varphi$ for some $k \in A$. Observe that, as x is an atom of A, $|[x] \vee [k]| \leqq 8$ by Lemma 1. Thus $|[b] \vee [t]| = |[x]\varphi \vee [k]\varphi| = |([x] \vee [k])\varphi| \leqq |[x] \vee [k]| \leqq 8$. Hence, by Lemma 1 again, either b or b', say b, is an atom of $A\varphi$. Let $\bar{x} = b$. We then have $[x]\varphi = [\bar{x}]$, proving Lemma 2.

REMARK. The correspondence $x \rightarrow \bar{x}$ is unambiguous if A has more than four elements.

We now associate with the map $\varphi: S(A) \rightarrow S(B)$ a map $\psi: A \rightarrow B$ in the following natural way. Let $a \in A$. Then $a = \bigvee (p_i \mid 1 \leqq i \leqq n)$, where for each $i = 1, \cdots, n$, p_i is an atom of A. Set $a\psi = \bigvee (\bar{p}_i \mid 1 \leqq i \leqq n)$.

It is obvious that ψ is a one-to-one Boolean homomorphism of A into B.

The following lemma is crucial.

LEMMA 3. *Assume the hypotheses of Lemma 2 and let $|A| > 4$. Then $[a\psi] = [a]\varphi$, for each $a \in A$.*

PROOF. If a is an atom of A, there is nothing to prove. Thus, assume that $a = p_1 \vee p_2 \vee \cdots \vee p_k$, where $k > 1$, and for each $i = 1, \cdots, k$, p_i is an atom of A; $p_i \neq p_j$ if $i \neq j$. Let $[a]\varphi = [\bar{a}]$. Our proof will be complete if we can show that the following holds:

$$(*) \qquad \begin{aligned} \text{either} \quad \bar{a} &= p_1\psi \vee p_2\psi \vee \cdots \vee p_k\psi, \\ \text{or} \quad \bar{a}' &= p_1\psi \vee p_2\psi \vee \cdots \vee p_k\psi. \end{aligned}$$

To prove this, suppose to the contrary that (*) is false. Then, since the $p_i\psi$ are atoms, we have that either (1) or (2) below hold.

(1) There is an s with $1 \leqq s < k$ such that $p_1\psi, \cdots, p_s\psi \leqq \bar{a}$ and $p_{s+1}\psi, \cdots, p_k\psi \leqq \bar{a}'$.

(2) $p_1\psi \vee \cdots \vee p_k\psi < \bar{a}$ or $p_1\psi \vee \cdots \vee p_k\psi < \bar{a}'$.

There are three cases:

Case 1. Either $p_1\psi \vee \cdots \vee p_s\psi = \bar{a}$ or $p_{s+1}\psi \vee \cdots \vee p_k\psi = \bar{a}'$.

By symmetry, it suffices to consider the former case. If $p_1\psi \vee \cdots \vee p_s\psi = \bar{a}$, then $p_s\psi$ is generated by $p_1\psi, \cdots, p_{s-1}\psi$, and $\bar{a}$. In fact,

$$p_s\psi = \bar{a} \wedge (p_1\psi \vee \cdots \vee p_{s-1}\psi)'.$$

Thus,

$$[p_s\psi] \subseteq [p_1\psi] \vee \cdots \vee [p_{s-1}\psi] \vee [\bar{a}].$$

Hence, by applying φ^{-1}, we have

$$\begin{aligned}[p_s] = [p_s\psi]\varphi^{-1} &\subseteq ([p_1\psi] \vee \cdots \vee [p_{s-1}\psi] \vee [a])\varphi^{-1} \\ &= [p_1] \vee \cdots \vee [p_{s-1}] \vee [a].\end{aligned}$$

Now p_s, being an atom of A, would be an atom of the subalgebra $[p_1] \vee \cdots \vee [p_{s-1}] \vee [a]$ of A. But the atoms of the latter are $p_1, \cdots, p_{s-1}$, a', and $a - (p_1 \vee \cdots \vee p_{s-1})$ $(= a \wedge p_1' \wedge \cdots \wedge p_{s-1}')$, which is a contradiction.

Case 2. $p_1\psi \vee \cdots \vee p_s\psi < \bar{a}$ and $p_{s+1}\psi \vee \cdots \vee p_k\psi < \bar{a}'$.

In this case, the subalgebra $[p_1\psi, \cdots, p_k\psi, \bar{a}]$ has $k+2$ atoms, namely, $p_1\psi, \cdots, p_k\psi$, $\bar{a} - (p_1\psi \vee \cdots \vee p_s\psi)$ and $\bar{a}' - (p_{s+1}\psi \vee \cdots \vee p_k\psi)$. Thus, by isomorphism, $[p_1, \cdots, p_k, a]$ has $k+2$ atoms, which is false, however, since $[p_1, \cdots, p_k, a]$ has, in fact, exactly $k+1$ atoms which are $p_1, \cdots, p_k$, and a'.

Case 3. $p_1\psi \vee \cdots \vee p_k\psi < \bar{a}$ or $p_1\psi \vee \cdots \vee p_k\psi < \bar{a}'$.

By using a similar argument as in Case 2, it can be shown that this case is again impossible.

Hence, we conclude that either $\bar{a} = p_1\psi \vee \cdots \vee p_k\psi$ or $\bar{a}' = p_1\psi \vee \cdots \vee p_k\psi$, which was to be shown.

Invoking Lemma 3, we arrive at the following result which provides important information on the relation between φ and ψ.

LEMMA 4. *Assume the hypotheses of Lemma* 3. *Let* $X \in S(A)$. *Then* $X\varphi = (X)\psi$.

PROOF. Let $x \in X$. Then, by Lemma 3 and the fact that φ is isotone, we have $x\psi \in [x\psi] = [x]\varphi \subseteq X\varphi$. Thus, $(X)\psi \subseteq X\varphi$.

Conversely, let $y \in X\varphi$. Then $[y] \subseteq X\varphi$. Since $X\varphi \in S(A)\varphi$ and $S(A)\varphi$ is a principal ideal of $S(B)$, we have $[y] \in S(A)\varphi$. Hence, we have $[y] = [x]\varphi$, for some $[x] \in S(A)$. As $[x]\varphi = [y] \subseteq X\varphi$ and φ is a one-to-one homomorphism, it follows that $[x] \subseteq X$. Thus, $x \in X$.

Now, we have, by Lemma 3 again, that $y \in [y] = [x]\varphi = [x\psi] \subseteq [X\psi] = X\psi$. Hence $X\varphi \subseteq (X)\psi$.

As a conclusion, we obtain the following result.

LEMMA 5. *The map* $\psi: A \to B$ *is uniquely determined by* $\varphi: S(A) \to S(B)$ *in case* $|A| \neq 4$.

PROOF. By Lemma 4, it suffices to prove the uniqueness part. For $|A|=2$ the assertion is trivial. If $\psi: A \to B$ is a one-to-one Boolean homomorphism inducing φ, then the image of an atom p of A is an atom of $A\psi = A\varphi$; moreover, $p\psi$ must belong to $[p]\varphi = \{0, 1, p\psi, (p\psi)'\}$, which contains at most one atom of $A\psi$ if $A\psi$ has at least eight elements. Hence the image of every atom of A under ψ is uniquely determined, and consequently so is the image of every element of A.

3. **Main result and applications.** We are now in a position to prove the main result.

THEOREM. *Let L be a lattice with property* (P), *and let $J(L)$ be the semilattice of its compact elements. Let α_a, for $a \in J(L)$, be an isomorphism: $(a] \to S(B(a))$ for some finite Boolean algebra $B(a)$. Let φ_{ab} for $a \leqq b$, a, $b \in J(L)$, be the mapping making the diagram of Figure* 1 *commutative where $i: (a] \to (b]$ is the inclusion map.*

$$\begin{array}{ccc} (a] & \xrightarrow{\alpha_a} & S(B(a)) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varphi_{ab}} \\ (b] & \xrightarrow[\alpha_b]{} & S(B(b)) \end{array}$$

FIGURE 1

Then there is a direct family $\mathscr{B} = \langle B(a), \psi_{ab} \rangle$ such that: (1) *ψ_{ab} induces φ_{ab}, i.e., $(X)\psi_{ab} = X\varphi_{ab}$ for any subalgebra $X \subseteq B(a)$*; (2) *the ψ_{ab} are uniquely determined for every $a \in J(L)$ of height $\neq 1$ (i.e., $|B(a)| \neq 4$)*; (3) *$S(B) \cong L$ where B is the direct limit of $\mathscr{B}$.*

PROOF. Since the diagram of Figure 1 is commutative for a, $b \in J(L)$, $a \leqq b$, $S(B(a))\varphi_{ab}$ is a principal ideal of $S(B(b))$. Hence, by previous lemmas in §2, there is a one-to-one Boolean homomorphism $\psi_{ab}: B(a) \to B(b)$ which induces φ_{ab}. By Lemma 5, the ψ_{ab} are uniquely determined if $|B(a)| \geqq 8$. From this and from the fact that $\varphi_{ab}\varphi_{bc} = \varphi_{ac}$, for $a \leqq b \leqq c$, it follows that $\psi_{ab}\psi_{bc} = \psi_{ac}$ if $a \leqq b \leqq c$ and $|B(a)| \neq 4$.

Since the uniqueness of ψ_{ab} inducing φ_{ab} fails for $a \in J(L)$ of height 1, we have to be a little careful in defining ψ_{ab} for a of height 1. Let a be an element of $J(L)$ of height 1. If there is no $b \in J(L)$ with $b > a$ then there is nothing to prove since then L is the lattice of subalgebras of a four element

Boolean algebra. Otherwise choose $\bar{a} \in J(L)$ covering a, i.e. $a<\bar{a}$, $\bar{a}$ is of height 2. Select $\psi_{a\bar{a}}:B(a)\to B(b)$ inducing $\varphi_{a\bar{a}}$ (there are, in fact, two possibilities for $\psi_{a\bar{a}}$). Let ψ_{aa} be the identity on $B(a)$. Now, for any $b>a$, let $c=b\vee\bar{a}$ and define ψ_{ab} such that the diagram of Figure 2 commutes.

$$\begin{array}{ccc} B(a) & \xrightarrow{\psi_{a\bar{a}}} & B(\bar{a}) \\ \psi_{ab}\downarrow & & \downarrow\psi_{\bar{a}c} \\ B(b) & \xrightarrow[\psi_{bc}]{} & B(c) \end{array}$$

FIGURE 2

Notice that $X=B(a)\psi_{a\bar{a}}\psi_{\bar{a}c}=B(a)\varphi_{ac}=a\alpha_c\subseteq B(c)$, and $Y=B(b)\psi_{bc}=B(b)\varphi_{bc}=b\alpha_c\subseteq B(c)$; hence $X\subseteq Y$ and thus ψ_{bc}^{-1} is defined on X, i.e. ψ_{ab} can indeed be determined in a unique way. It easily follows that ψ_{ab} induces φ_{ab}. Having defined ψ_{ab} for any $a\leqq b$ in $J(L)$, the transitivity $\psi_{ab}\psi_{bc}=\psi_{ac}$ is easily checked for any $a\leqq b\leqq c$ in $J(L)$. Since $J(L)$ is a join semilattice and hence is a directed set, we have a direct family $\mathscr{B}=\langle B(a), \psi_{ab}\rangle$ of finite Boolean algebras. Let B be the direct limit of $\mathscr{B}$. We shall prove that $L\cong S(B)$. Since L is generated by $J(L)$, to define a map from L to $S(B)$, we define $f:J(L)\to S(B)$ as follows: for $x \in J(L)$ set $f(x)=B(x)\psi_{x\infty}$.

Note that as ψ_{ab} is a one-to-one Boolean homomorphism for all $a\leqq b$, $a, b \in J(L)$, $\psi_{x\infty}$ is a one-to-one Boolean homomorphism for each $x \in J(L)$.

We claim that, if $x, y \in J(L)$, then $x\leqq y$ if and only if $B(x)\psi_{x\infty}\subseteq B(y)\psi_{y\infty}$.

To see this, let $x, y \in J(L)$. Then there exists $u \in J(L)$ such that $x, y\leqq u$. By means of the commutative diagram of Figure 1, it follows that $x\alpha_x\varphi_{xu}=xi\alpha_u$, i.e.,

$$x\alpha_u = B(x)\varphi_{xu} = B(x)\psi_{xu},$$

by Lemma 5. Similarly,

$$y\alpha_u = B(y)\psi_{yu}.$$

Now, observe that $x\leqq y$:

if and only if $x\alpha_u\subseteq y\alpha_u$,

if and only if $B(x)\psi_{xu}\subseteq B(y)\psi_{yu}$,

if and only if $B(x)\psi_{xu}\psi_{u\infty}\subseteq B(y)\psi_{yu}\psi_{u\infty}$,

if and only if $B(x)\psi_{x\infty}\subseteq B(y)\psi_{y\infty}$,

which was to be shown.

Hence, f is a one-to-one order- and inverse order-preserving map.

Let A be a finite subalgebra of B. Then there is $u \in J(L)$ such that A is a subalgebra of $B(u)\psi_{u\infty}$. Thus, $A\psi_{u\infty}^{-1}$ is a subalgebra of $B(u)$. Hence $A\psi_{u\infty}^{-1}=B(x)\psi_{xu}$ for some $x \in J(L)$, $x\leqq u$. Observe that

$$f(x) = B(x)\psi_{x\infty} = B(x)\psi_{xu}\psi_{u\infty} = A\psi_{u\infty}^{-1}\psi_{u\infty} = A.$$

Therefore, we conclude that f is an isomorphism of $J(L)$ onto the semilattice of compact elements of $S(B)$ and hence can be extended to an isomorphism of L onto $S(B)$. The proof of the theorem is thus complete.

The following are two applications of the main result.

COROLLARY 1. *The lattice $S(B)$ of all subalgebras of a Boolean algebra B can be characterized as a lattice satisfying property* (P).

COROLLARY 2 (D. SACHS). *The lattice $S(B)$ determines the Boolean algebra B up to isomorphism.*

Corollary 1 follows immediately from the main theorem. To prove Corollary 2, let us say that a direct family $\langle B_a, \psi_{ab}\rangle$, $a \leqq b$, $a, b \in J(L)$, is associated with L if it satisfies the conclusion of the theorem with some system (α_a), $a \in J(L)$, of isomorphisms $\alpha_a : (a] \to S(B_a)$. It easily follows from the uniqueness part of the theorem that any two direct families associated with L, and hence their direct limits too, are isomorphic. Now, let $L = S(A)$ for some Boolean algebra A. Consider the inclusion maps $\psi_{ab} : a \to b$ for $a \subseteq b$, a, b subalgebras of A. Then the system $\langle a, \psi_{ab}\rangle$ is associated with L, in fact with the α_a being identity mappings. Moreover, the direct limit of $\langle a, \psi_{ab}\rangle$ is isomorphic to A. As a conclusion, we have that for $L \cong S(A)$, A a Boolean algebra, some direct family associated with L has a direct limit isomorphic to A. If we have $S(A) \cong S(B)$, then the last assertion applied twice to $L = S(A)$ gives direct families associated with L with direct limits isomorphic to A and B, respectively, hence $A \cong B$.

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