PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, Number 1, November 1972

CONSISTENCY OF LINEAR INEQUALITIES OVER SETS

ABRAHAM BERMAN

ABSTRACT. Necessary and sufficient conditions for the equivalence of the statements: (I) The system $b-Ax\in T$, $x\in S$, is consistent. (II) $y\in T^*$, $A^Hy\in S^*\Rightarrow \mathrm{Re}(b,y)\geqq 0$, are given in terms of the sets S and T and the matrix A. Sufficient conditions for this equivalence are obtained in the case where S and T are closed convex cones.

Introduction. A main tool in the theory of symmetric linear programming (e.g. [3]) is the following theorem:

Let $S\subset C^n$ and $T\subset C^m$ be polyhedral cones and let $A\in C^{m\times n}$. Then

(I) The system $b-Ax\in T$, $x\in S$, is consistent if and only if

(II) $y\in T^*$, $A^Hy\in S^*\Rightarrow \mathrm{Re}(b,y)\geqq 0$.

This equivalence does not hold for general cones, that is, there are nonpolyhedral cones for which (I) and (II) are not equivalent. On the other hand, there are sets S and T which are not cones such that for a suitable A, (I) and (II) are equivalent.

The main purpose of this paper is to characterize those triplets (S, T, A) for which (I) and (II) are equivalent for every $b\in C^n$. This is done in §2 which follows a section of notation and preliminaries. The case where S and T are closed convex cones is described in the third section where sufficient conditions, including results of Ben-Israel [1], Fan [4] and Sposito-David [8], for the equivalence of (I) and (II) are derived as corollaries.

1. **Notation and preliminaries.** C^n denotes the n-dimensional complex vector space. R^n denotes the n-dimensional real vector space. R^n_+ is the nonnegative orthant in R^n. $C^{m\times n}$ denotes the space of $m\times n$ complex matrices.

For $x, y\in C^n$: $\mathrm{Re}\, x$ is the real part of x,

$$(x,y)=\overline{(y,x)} \text{ is the inner product of } x \text{ and } y.$$

Presented to the Society, July 8, 1971; received by the editors July 28, 1971.

AMS 1969 *subject classifications.* Primary 1550, 5230; Secondary 5210, 9050.

Key words and phrases. Affine set, closed convex cone, consistency, linear inequality, linear transformation, pointed cone, polar, polyhedral cone, relative interior.

For S, $T \subset C^n$: $S \times T$ is the cartesian product.

I_n is the identity matrix of order n.

For $A \in C^{m\times n}$: A^H is the conjugate transpose of A, $R(A)$ is the range of A, $N(A)$ is the null space of A, $[A, I] = [A, I_m]$, $N[A, I] = \{[x, y] \in C^n \times C^m, y = -Ax\}$.

A nonempty set $S \subset C^n$ is

(i) *convex* if $0 \leqq \lambda \leqq 1 \Rightarrow \lambda S + (1-\lambda)S \subset S$,

(ii) a *cone* if $0 \leqq \lambda \Rightarrow \lambda S \subset S$,

(iii) a *convex cone* if it satisfies (i) and (ii),

(iv) a *pointed cone* if it satisfies (i) and $S \cap (-S) = 0$,

(v) a *polyhedral cone* if $S = BR^k_+$, for some $B \in C^{n\times k}$.

A polyhedral cone is a closed convex cone. The *polar*, S^*, of S is given by

$$S^* = \{y \in C^n \mid x \in S \Rightarrow \operatorname{Re}(x, y) \geqq 0\}.$$

If $S \subset T$, then $S^* \supset T^*$. The polar of a subspace is its orthogonal complement. In particular $(N(A))^* = R(A^H)$. S^* is a closed convex cone. $S^{**} = (S^*)^*$ is the smallest closed convex cone which contains S. Thus, $S^{**} = S$ if and only if S is a closed convex cone, and $S^* = S^{***}$.

The interior of S^*, int S^*, is given algebraically by

$$\operatorname{int} S^* = \{y \in S^* \mid 0 \neq x \in S \Rightarrow \operatorname{Re}(x, y) > 0\}.$$

For a cone S, int $S^* \neq \varnothing$ if and only if S is pointed.

A set $S \subset C^n$ is *affine* if $(1-\lambda)S + \lambda S \subset S$ for all real λ. Every affine set is a translation of a subspace. The smallest affine set containing S, denoted by aff S, is called the *affine hull* of S. When S is a convex cone, aff S is a subspace.

The *relative interior* of a convex set S, denoted by ri S, is the interior which results when S is regarded as a subset of aff S. When S is a pointed cone, ri $S^* = \operatorname{int} S^*$. For more on cones, affine sets and relative interiors the reader is referred to [7], especially §§1 and 6.

Let S be a closed convex cone in C^n, $A \in C^{m\times n}$, $\bar{S}$ the closed convex cone S when regarded as a subset of aff S, and $\bar{S}^*$ the polar cone of $\bar{S}$ (in aff S). Decompose C^n to aff $S \times (\text{aff } S)^*$ and define $\bar{A}$, a linear transformation from aff S to C^m, by $\bar{A}x = A[x, \bar{0}]$ where $\bar{0}$ is the origin of $(\text{aff } S)^*$. Then ri $S = \operatorname{int} \bar{S}$, $S^* = (\bar{S})^* \times (\text{aff } S)^*$ and $\bar{A}^H z$ is the perpendicular projection of $A^H z$ on aff S.

2. **Consistency over sets.** Triplets S, T, A for which (I) and (II) are equivalent are characterized in the following theorem.

THEOREM 1. *Let $A \in C^{m\times n}$, $S\subset C^m$ and $T\subset C^m$. Then the following are equivalent*:

(a) (I) *and* (II) *are equivalent for every* $b \in C^n$.

(b) $AS+T$ *is a closed convex cone and* $AS\cup T\subset AS+T$.

(c) $N(A, I)+S\times T$ *is a closed convex cone and* $AS\cup T\subset AS+T$.

PROOF. First notice that

$$A^H y \in S^* \Leftrightarrow y \in (AS)^* \tag{1}$$

since both sides of (1) are equivalent to $x \in S \Rightarrow \mathrm{Re}(Ax, y) \geqq 0$. From (1) it follows that (II)$\Leftrightarrow[y \in (AS)^* \cap T^* \Rightarrow \mathrm{Re}(b, y) \geqq 0] \Leftrightarrow b \in ((AS)^* \cap T^*)^*$, while (I)$\Leftrightarrow b \in AS+T$. Thus (a) may be rewritten as

(a′) $AS+T=((AS)^*\cap T^*)^*$.

We shall show that (a′)$\Leftrightarrow$(b)$\Leftrightarrow$(c).

(a′)$\Rightarrow$(b). $AS+T$ is a closed convex cone since it is a polar of a set. Taking polars of the two sides of (a′) one gets

$$(AS + T)^* = ((AS)^* \cap T^*)^{**} = (AS)^* \cap T^*,$$

since the intersection of two closed convex cones is a closed convex cone. Thus $(AS+T)^*\subset(AS)^*$ and $(AS+T)^*\subset T^*$, implying $AS\subset AS^{**}\subset (AS+T)^{**}=AS+T$ and $T\subset T^{**}\subset(AS+T)^{**}=AS+T$ so that $AS\cup T\subset AS+T$.

(b)$\Rightarrow$(a′). It is always true that $(AS)^*\cap T^*\subset(AS+T)^*$. Also $AS\cup T\subset AS+T \Rightarrow (AS+T)^*\subset(AS)^*\cap T^*$. Thus (b)$\Rightarrow (AS)^*\cap T^*=(AS+T)^*$ and $((AS)^*\cap T^*)^*=(AS+T)^{**}=AS+T$, since $AS+T$ is a closed convex cone.

(b)$\Leftrightarrow$(c). Since the last parts of the conditions are identical, one has to show that $AS+T$ is a closed convex cone (in C^m) if and only if $N[A, I]+S\times T$ is a closed convex cone (in $C^{m\times n}$).

Since $AS+T=[A, I](S\times T)$ it suffices to show that $N(A)+S$ is a closed convex cone if and only if AS is one. This is an unpublished result of Robert A. Abrams. The proof here seems to be new. First, notice that $N(A)+S$ is the (unique) largest set whose image under A is AS, that is

$$Ax \in AS \Leftrightarrow x \in N(A) + S. \tag{2}$$

We now show that

$$Ax \in (AS)^{**} \Leftrightarrow x \in (N(A) + S)^{**}. \tag{3}$$

To do this we notice that

$$Ax \in (AS)^{**} \Leftrightarrow x \in (A^H(AS)^*)^* \Leftrightarrow x \in (S^* \cap R(A^H))^*,$$

the two equivalences following from (1). Thus in order to prove (3) it suffices to show that $(N(A)+S)^*=R(A^H)\cap S^*$. It is clear that $R(A^H)\cap S^*\subset(N(A)+S)^*$. To show the inclusion in the other direction let $y\in(N(A)+S)^*$. Then $z\in N(A)$, $\omega\in S$, $\lambda\in R\Rightarrow\operatorname{Re}(\lambda z+\omega, y)\geqq 0$.

As λ can take any sign, $\operatorname{Re}(\lambda z+\omega, y)=\lambda\operatorname{Re}(z, y)+\operatorname{Re}(\omega, y)\geqq 0$ can only hold if $\operatorname{Re}(z, y)=0$. Thus $y\in(N(A))^*=R(A^H)$.

Also $0\in N(A)\Rightarrow S\subset N(A)+S\Rightarrow(N(A)+S)^*\subset S^*$, which completes the proof of (3). Now the equivalence of (b) and (c) follows from (2), (3) and the fact that AS and $(AS)^{**}$ are contained in $R(A)$.

3. **Consistency over cones.** It is clear that S and T which satisfy the conditions of Theorem 1 do not have to be cones. However in most cases of interest in the system of (I), S and T are closed convex cones and this is what we assume in this section.

THEOREM 2. *Let $A\in C^{m\times n}$ and let $S\subset C^n$, $T\subset C^m$ be closed convex cones. Then any of the following conditions is sufficient for any of the conditions of Theorem* 1.

(a) *$N(A, I)+S\times T$ (or $AS+T$) is closed.*

(b) *$N(A, I)\cap S\times T$ is a subspace.*

(c) *S is pointed and the system $y\in T^*$, $A^Hy\in\operatorname{int} S^*$, is consistent.*

(d) *S and T are pointed and the system $y\in\operatorname{int} T^*$, $A^Hy\in\operatorname{int} S^*$, is consistent.*

(e) *S and T are polyhedral cones.*

PROOF. The condition that $AS\cup T\subset AS+T$ is satisfied since S and T are cones and thus $0\in AS\cap T$. Condition (a) is stated for completeness. We show (d)$\Rightarrow$(c)$\Rightarrow$(b)$\Rightarrow$(a)$\Leftarrow$(e).

(d)$\Rightarrow$(c). Since A^H is continuous, (c) is equivalent to

(c′) S is pointed and the system $y\in\operatorname{ri} T^*$, $A^Hy\in\operatorname{int} S^*$, is consistent. If now T is pointed, then (d) and (c) are the same.

(c)$\Rightarrow$(b). Regarding T^* as a subset of aff T^*, (c′) becomes

(c″) The system $y\in\operatorname{int}\bar{T}^*$, $\bar{A}^Hy\in\operatorname{int} S^*$, is consistent.

By Corollary 1.4 of [2], this is equivalent to

$$-(\bar{A}^H)^Hz\in(\bar{T}^*)^*,\quad z\in S\Rightarrow z=0,$$

which may be rewritten as: $(-(\bar{A}^H)^Hz, t)\in(\bar{T}^*)^*\times(\operatorname{aff}\bar{T}^*)$, $z\in S\Rightarrow z=0$, or $-Az\in T$, $z\in S\Rightarrow z=0$. Thus, $N[A, I]\cap S\times T=0$ which implies (b).

(b)$\Rightarrow$(a). Follows from Lemma 1.2 of [2] by replacing A by $[A, I]$ and S by $S\times T$.

(e)$\Rightarrow$(a). The image, by a linear transformation, of a polyhedral cone is a polyhedral cone. The sum of two polyhedral cones is a polyhedral cone. So $AS+T$ is a polyhedral cone and thus a closed convex cone.

The paper is concluded with two remarks on Theorem 2.

1. Conditions (a) and (e) of Theorem 2, were proved, in the case where T is the origin, by Ben-Israel [**1**, Theorems 2.4 and 3.5]. Special choices of S include the Lemma of Farkas [**5**] (where $S=R^n_+$) and a theorem of Levinson [**6**]. Condition (c) of Theorem 2 is due to Sposito-David [**8**]. Condition (d) is a finite version of Theorem 5 of Fan [**4**].

2. The conditions in Theorem 2 are not equivalent. To show this let $T=C^m$ or $A=0$. In both cases (c) cannot hold and (a) holds, while (b) is true if and only if S is a subspace.

The author wishes to thank the referee for his helpful remarks.

References

1. A. Ben-Israel, *Linear equations and inequalities on finite dimensional, real or complex, vector spaces: A unified theory*, J. Math. Anal. Appl. **27** (1969), 367–389. MR **39** #4192.

2. A. Berman and A. Ben-Israel, *More on linear inequalities with applications to matrix theory*, J. Math. Anal. Appl. **33** (1971), 482–496.

3. ———, *Linear inequalities, mathematical programming and matrix theory*, Math. Prog. **1** (1971), 291–300.

4. K. Fan, *A generalization of the Alaoglu-Bourbaki theorem and applications*, Math. Z. **88** (1965), 48–60. MR **31** #2584.

5. J. Farkas, *Über Theorie der einfachen Ungleichungen*, J. Reine Angew. Math. **124** (1902), 1–24.

6. N. Levinson, *Linear programming in complex space*, J. Math. Anal. Appl. **14** (1966), 44–62. MR **37** #1162.

7. R. T. Rockafellar, *Convex analysis*, Princeton Math. Series, no. 28, Princeton Univ. Press, Princeton, N.J., 1970. MR **43** #445.

8. V. A. Sposito and H. T. David, *A note on Farkas lemmas over cone domains*, Iowa State University Report.

Centre de Recherches Mathématiques, Université de Montréal, Montréal, Québec, Canada

Current address: Department of Mathematics, Technion–Israel Institute of Technology, Haifa, Israel