VALENTINE'S EXTENSIONS OF TIEzte's
THEOREM ON CONVEX SETS

NICK M. STAVRAKAS AND R. E. JAMISON

Abstract. New proofs are given for Valentine's extensions of Tietze's theorem on convex sets.

Valentine in [1] proved the following two extensions of Tietze's theorem on convex sets.

Theorem 1. Let S be a closed connected subset of $\mathbb{R}^d$ such that S has at most n points of local nonconvexity. Then S is an $L^{n+1}$ set.

Theorem 2. Let S be a closed connected subset of $\mathbb{R}^d$ such that the points of local nonconvexity of S can be decomposed into n convex sets. Then S is an $L^{2n+1}$ set.

The purpose of this paper is to give new proofs of these theorems. The proofs depend upon the following two lemmas.

Lemma 1. Let S be a subset of $\mathbb{R}^d$. Let $x, y \in S$ and suppose I is an arc from x to y of minimal arc length in S such that each point of I is a point of local convexity of S, except possibly for x and y. Then $I = [xy]$, the closed line segment from x to y.

Lemma 2. Let S be a closed connected subset of $\mathbb{R}^d$, which has at least one point of local nonconvexity. Then given $x \in S$ there exists a point p of local nonconvexity of S such that $[xp] \subseteq S$.

The proof of Lemma 1 is easy and is omitted. The proof of Lemma 2 is in Valentine [1]. We only prove Theorem 2, and then it will be clear how to prove Theorem 1 using the same method.

Proof of Theorem 2. Let $\{C_1, \cdots, C_n\}$ be the n convex sets into which the points of local nonconvexity can be decomposed. Now let $z \in S$. Define $S(z)$ as $\{t : t \in S$ and $[zt] \subseteq S\}$ and define $S(C_i) as \bigcup_{z \in C_i} S(z)$. Clearly $S(C_i)$ is an $L_3$ set. Lemma 2 implies $S = \bigcup_{i=1}^{n} S(C_i)$. The fact that S is connected then implies that S is an $L_{2n}$ set. Thus, given $x, y \in S$, there exists an arc from x to y in S of finite arc length, and hence there exists an

Received by the editors November 24, 1971.

AMS 1970 subject classifications. Primary 52A20; Secondary 52A15.

© American Mathematical Society 1972

229
arc from $x$ to $y$ of minimal arc length in $S$. Let $l$ be such an arc. $l$ is clearly simple. The minimality of $l$ and the convexity of $C_i$ imply that $l \cap \bar{C}_i$ is a point or closed line segment or empty. Remove the points of local non-convexity of $S$ from $l$ to get a set $l'$. Note that we remove at most $n$ sets, each of which must be a closed line segment or a point. Thus $l'$ has at most $n+1$ components, and the closure of each of these components is an arc satisfying the hypothesis of Lemma 1, and hence is a line segment. Thus $l$ consists of at most $2n+1$ line segments and the theorem follows.

Bibliography


Department of Mathematics, University of North Carolina, Charlotte, North Carolina 28205 (Current address of Nick M. Stavrakas)

Current address (R. E. Jamison): Department of Mathematics, University of Washington, Seattle, Washington 98195