SEPARATION BY CYLINDRICAL SURFACES

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Abstract. C. Carathéodory has established that two compact sets \( P \) and \( Q \) in Euclidean \( n \)-space can be strictly separated by a hyperplane if each subset of \( n + 1 \) or fewer points of \( Q \) can be strictly separated from \( P \) by a hyperplane. In this paper it is shown that if each subset of \( k \) or fewer points of \( Q \) can be strictly separated from \( P \) by a hyperplane (where \( k \) is a fixed integer, \( 1 \leq k \leq n \)), then there exists a cylinder of an appropriate sort containing \( P \) and disjoint from \( Q \).

In 1905, C. Carathéodory [1] established that two compact sets \( P \) and \( Q \) in Euclidean \( n \)-space \( E^n \) can be strictly separated by a hyperplane if each subset of \( n + 1 \) or fewer points of \( Q \) can be strictly separated from \( P \) by a hyperplane. Suppose, however, that less information is known. Specifically, suppose for a fixed integer \( k \) (\( 1 \leq k \leq n \)) that each subset of \( k \) or fewer points of \( Q \) can be strictly separated from \( P \) by a hyperplane. Then what can be said about the type of separation possible between \( P \) and \( Q \)? A partial answer to this question for the case where \( P \) consists of a single point and \( k = n \) was given by Hanner and Rådström [2] in the following theorem.

Theorem 1. Let \( Q \) be a compact subset of \( E^n \) and let \( p \) be a fixed point. Suppose that for every \( n \) points of \( Q \) there exists a hyperplane strictly separating those \( n \) points from \( p \). Then there exists a hyperplane containing \( p \) which is disjoint from \( Q \).

In order to generalize this result to the case where \( P \) is an arbitrary compact set, the following concept is useful.

Definition 1. Let \( A \) be a nonempty compact subset of \( E^n \) and let \( F \) be a \( k \)-dimensional subspace of \( E^n \) \( (0 \leq k \leq n) \). Then \( C = A + F = \{a + f : a \in A \text{ and } f \in F\} \) is called the \( k \)-cylinder generated by \( A \) and \( F \).
Notice that the 0-cylinder generated by \( A \) is precisely \( A \). The 1-cylinder generated by \( A \) and a line through the origin is just a cylinder in the sense of the usual definition. An \((n-1)\)-cylinder is a pair of parallel supporting hyperplanes and the region between them. The \( n \)-cylinder generated by \( A \) is precisely \( E^n \).

We are now in a position to state our main result. We shall denote the convex hull of a set \( A \) in \( E^n \) by \( \text{conv} \ A \).

**Theorem 2.** Let \( P \) and \( Q \) be nonempty compact subsets of \( E^n \). Suppose for a fixed integer \( k \) \((1 \leq k \leq n)\) that each subset of \( k \) or fewer points of \( Q \) can be strictly separated from \( P \) by a hyperplane. Then given any \( k \)-cylinder of the form \( C = (\text{conv} \ P) + F \) there exists a \((k-1)\)-cylinder of the form \( D = (\text{conv} \ P) + F \), such that \( D \subseteq C \) and \( D \cap Q = \emptyset \).

It is clear that the Hanner-Rådström result is included in Theorem 2 as the special case where \( P \) is a single point and \( k = n \), for then the \( k \)-cylinder generated by \( P \) is just \( E^n \) and the \((k-1)\)-cylinder containing \( P \) and disjoint from \( Q \) is precisely a hyperplane. Unfortunately the approach used by Hanner and Rådström does not adapt itself to the situation where \( P \) is larger than a single point. In order to prove Theorem 2 in its full generality we recall the following definition and theorems relating to spherical convexity.

**Definition 2.** A subset \( K \) of a spherical surface in \( E^n \) is said to be strongly convex if \( K \) does not contain antipodal points and if \( K \) contains, with each pair of its points, the small arc of the great circle determined by them.

**Theorem 3.** Let \( \Omega \) be the unit sphere about the origin in \( E^n \), and let \( \mathcal{A} = \{ A_i : i \in I \} \) be a family of compact strongly convex subsets of \( \Omega \). If each \( n \) or fewer members of \( \mathcal{A} \) have a point in common, then there exists a pair of antipodal points \( \{y, -y\} \) in \( \Omega \) such that each \( A_i \) (\( i \in I \)) intersects \( \{y, -y\} \).

**Proof.** See Horn [4].

**Theorem 4.** Let \( \Omega \) be the unit sphere about the origin in \( E^n \), and let \( \mathcal{A} = \{ A_i : i \in I \} \) be a family of compact strongly convex subsets of \( \Omega \). If each \( n+1 \) or fewer members of \( \mathcal{A} \) have a point in common, then there exists a point in common to all the members of \( \mathcal{A} \).

**Proof.** This follows directly from Helly's theorem [3].

**Proof of Theorem 2.** Letting \( d(a, b) = \|a - b\| \) for \( a, b \in E^n \), we denote \( \inf\{d(a, b) : a \in A, b \in B\} \) by \( d(A, B) \) for subsets \( A \) and \( B \) of \( E^n \). Define \( \delta = \inf\{d(\text{conv} \ T, \text{conv} \ P) : T \) is a subset of \( k \) or fewer points of \( Q \} \). Since \( P \) and \( Q \) are compact it follows that \( \delta > 0 \). Given a \( k \)-cylinder \( C = (\text{conv} \ P) + F \), if \( Q \cap C = \emptyset \) then the result follows directly. If \( Q \cap C \neq \emptyset \), then we let
\( \Omega \) be the intersection of \( F \) and the unit sphere about the origin in \( E^n \). For each point \( w \in \Omega \) we define \( r_w \) to be the ray from the origin through \( w \) and \( F_w \) to be the \((k-1)\)-dimensional subspace contained in \( F \) which is perpendicular to \( r_w \). Then for each point \( q \in Q \cap C \), we define

\[
A_q \equiv \{ w \in \Omega : S_q \text{ is contained in the component of } C \sim [(\text{conv } P) + F_w] \text{ which intersects } (\text{conv } P) + r_w \}
\]

where \( S_q \equiv \{ x : \| x - q \| < \delta / 2 \} \). (See Figure 1.)

![Figure 1](image)

We claim first that, for each \( q \in Q \cap C \), \( A_q \) is a compact strongly convex subset of \( \Omega \). To see this we pick an \( A_q \) and define, for each \( w \in A_q \), \( S_w \) to be the component of \( C \sim [(\text{conv } P) + F_w] \) which intersects \( (\text{conv } P) + r_w \). Thus given two distinct points \( w \) and \( w' \) in \( A_q \) we must show that \( q \in S_x \) where \( x \) is a point on the small arc of the great circle determined by \( w \) and \( w' \). Now \( q \in S_w \cap S_{w'} \) and \( S_w \cap S_{w'} \) is contained in \( S_x \), so \( q \in S_x \). Furthermore, since \( S_q \) is open, \( A_q \) is a compact subset of some open hemisphere and thus contains no antipodal points.

Secondly, we claim that if \( q_1, \ldots, q_m \) \((1 \leq m \leq k)\) are any \( m \) points in \( Q \cap C \), then \( \bigcap_{i=1}^{m} A_{q_i} \neq \emptyset \). To see this we note that \( d(\text{conv}\{q_1, \ldots, q_m\}, \text{conv } P) \geq \delta \). Thus \( d(\text{conv}\{S_{q_1}, \ldots, S_{q_m}\}, \text{conv } P) > \delta / 2 \) and so there exists a hyperplane \( H \) strictly separating \( \{S_{q_1}, \ldots, S_{q_m}\} \) and \( P \). Let \( H' \) be the \((n-1)\)-dimensional subspace parallel to \( H \), and let \( G \equiv H' \cap F \). Now \( F \nparallel H' \) since
$H$ separates $P$ and $\{S_{q_1}, \cdots, S_{q_n}\}$, and so $G$ is a $(k-1)$-dimensional subspace. It follows that $\{S_{q_1}, \cdots, S_{q_m}\}$ is contained in one of the two components of $C \setminus [(\text{conv } P)+G]$, and we may choose $w \in \Omega$ so that $r_w$ is perpendicular to $G$ and $(\text{conv } P)+r_w$ intersects the component that contains $\{S_{q_1}, \cdots, S_{q_m}\}$. Then $w \in \bigcap_{i=1}^{n} A_{q_i}$.

Combining the two claims and Theorem 3, we see that there exists a pair of antipodal points $\{y, -y\}$ in $\Omega$ such that each $A_q$ ($q \in Q \cap C$) intersects $\{y, -y\}$. It follows that the $(k-1)$-cylinder $(\text{conv } P)+F_v$ has empty intersection with $Q$.

**Example.** To see that Theorem 2 is a best theorem in the sense that one cannot in general weaken the hypotheses and obtain the same conclusion, let $P$ be a circle in $E^2$ and let $Q = \{q_1, q_2, q_3\}$ be the vertices of a triangle such that $P$ intersects each edge of the triangle but each vertex is outside $P$. (See Figure 2.) Then each point of $Q$ can be strictly separated from $P$ by a hyperplane, but clearly there does not exist any 1-cylinder generated by $P$ which misses $Q$.

Theorem 2 is not a proper generalization of Carathéodory's theorem, but by changing the conditions slightly we obtain the following theorem which includes Carathéodory's result.

**Theorem 5.** Let $P$ and $Q$ be nonempty compact subsets of $E^n$. Suppose for a fixed integer $k$ ($2 \leq k \leq n+1$) that each subset of $k$ or fewer points of $Q$ can be strictly separated from $P$ by a hyperplane. Then given any $(k-1)$-cylinder of the form $D = (\text{conv } P) + F_1$ there exists a $(k-2)$-cylinder of the form $E = (\text{conv } P) + F_2$ such that $E \subset D$ and $Q \cap D$ is contained in one of the two connected components of $D \sim E$.

**Proof.** Let $\delta = \inf \{d(\text{conv } T, \text{conv } P) : T \text{ is a subset of } k \text{ or fewer points of } Q\}$. Since $P$ and $Q$ are compact it follows that $\delta > 0$. Given a $(k-1)$-cylinder $D = (\text{conv } P) + F_1$, if $Q \cap D = \emptyset$ then the result follows directly. If $Q \cap D \neq \emptyset$, then we let $\Omega$ be the intersection of $F_1$ and the unit sphere about the origin in $E^n$. For each point $w \in \Omega$ we define $r_w$ to be the ray from the origin through $w$ and $F_w$ to be the $(k-2)$-dimensional subspace contained in $F_1$ which is perpendicular to $r_w$. Then for each point $q \in Q \cap D$,
we define

\[ A_q \equiv \{ w \in \Omega : S_q \text{ is contained in the component of} \]
\[ D \sim [(\text{conv } P) + F_w] \text{ which intersects } (\text{conv } P) + r_w \]

where \[ S_q \equiv \{ x : \| x - q \| < \delta / 2 \} \]. (See Figure 1.)

As in the proof of Theorem 2, we see that each \( A_q \) (\( q \in Q \cap D \)) is a compact strongly convex subset of \( \Omega \) and that if \( q_1, \ldots, q_m \) (\( 1 \leq m \leq k \)) are any \( m \) points in \( Q \cap D \), then \( \bigcap_{q \in Q \cap D} A_q \neq \emptyset \). Since \( \Omega \) is the unit sphere about the origin in the \( (k-1) \)-dimensional space \( F_1 \), Theorem 4 implies that there exists a point \( y \) in \( \bigcap_{q \in Q \cap D} A_q \). Letting \( E \equiv (\text{conv } P) + F_y \) we have \( Q \cap D \) contained in the component of \( D \sim E \) which intersects \( (\text{conv } P) + r_y \).

It should be noted that when \( k = n+1 \), then Theorem 5 is equivalent to the following theorem due to P. Kirchberger [5]. A proof of this equivalence may be found in Lay [6, p. 32].

**Theorem 6.** Let \( P \) and \( Q \) be nonempty compact subsets of \( E^n \). Then \( P \) and \( Q \) can be strictly separated by a hyperplane if and only if for each subset \( T \) of \( n+2 \) or fewer points of \( P \cup Q \) there exists a hyperplane which strictly separates \( T \cap P \) and \( T \cap Q \).

**References**


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