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SELF-UNIVERSAL CRUMPLED CUBES AND A DOGBONE SPACE

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ABSTRACT. The question of whether each self-universal crumpled cube is universal is answered negatively by presenting an example of a dogbone space which is not topologically E^3 but which can be expressed as a sewing of two crumpled cubes, one of which is self-universal.

C. D. Bass and R. J. Daverman [2] presented a brief paper indicating that the solid Alexander horned sphere is an example of a crumpled cube which is self-universal but not universal, thus answering negatively the question asked by C. E. Burgess and J. W. Cannon in [4] of whether each self-universal crumpled cube is universal. The validity of the example presented by Bass and Daverman depends on a claim that a certain upper semicontinuous decomposition of S^3 into points and tame arcs, described in [2], is not topologically S^3, which in turn depends on the validity of four lemmas which are stated in [2, §2]. The proofs of these four lemmas appear to entail nontrivial arguments which are not included in [2].

In this note, we present an example of a dogbone space, an upper semicontinuous decomposition of S^3 into points and tame arcs whose nondegenerate elements can be expressed as the intersection of a tower of solid double tori, which is not topologically S^3. The dogbone space can be described as the result of a sewing of two crumpled cubes, one of which is self-universal. Thus, the question asked by Burgess and Cannon in [4] is answered negatively. The argument will be based essentially upon work by Casler [5] and the author [1].

Some recent work by Eaton [6] includes a different proof that the solid Alexander horned sphere H, used in the example presented by Bass and Daverman, is not universal. This was done by sewing H to the crumpled cube T described by Stallings [7] so that the wild points of Bd H are sewn to the Cantor set of nonpiercing points of T. Other methods developed by Eaton, in papers cited in [6], should offer alternative ways

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to show that neither the decomposition space we describe here nor the one described by Bass and Daverman [2] is topologically S^3.

Definitions relating to crumpled cubes will be those of [2].

1. **The example.** The description of the dogbone space of this paper is modeled after that by Bing in [3]. As in Figure 1, let H_1 and H_2 denote

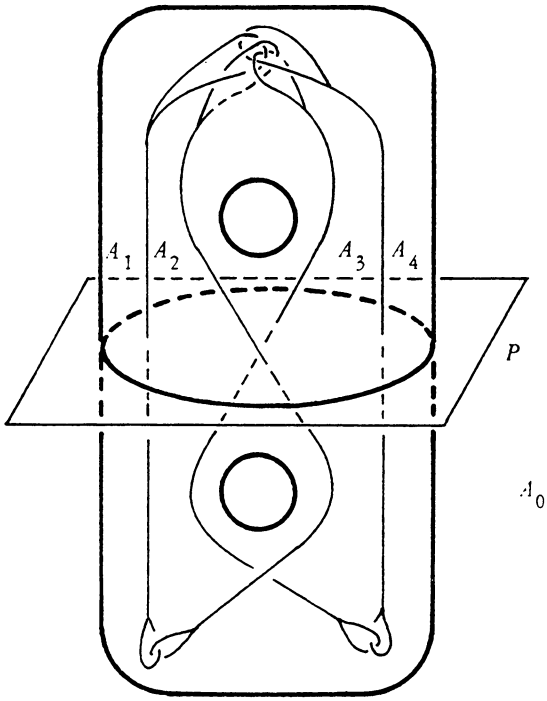

FIGURE 1

the upper and lower half-spaces of S^3, and P the xy-plane. Let A_0 denote a solid double torus which intersects P in a disk. Then, embed solid double tori $A_1, \cdots, A_4$, linked as indicated, in A_0; although each of $A_1, \cdots, A_4$ is shown as a finite graph, it is topologically equivalent to A_0. Then, for each $i=1, \cdots, 4$, solid double tori $A_{i,1}, \cdots, A_{i,4}$ are embedded in A_1 such that there is a homeomorphism of S^3 onto itself which is the identity on the complement of some open set containing A_0 and takes A_0 onto A_i. Succeeding steps of the construction are to be described inductively.

Let $M=A_0 \cap \sum A_i \cap \sum A_{i,j} \cap \sum A_{i,j,k} \cdots$. Let G be the set whose elements are components of M and one-point subsets of S^3-M. Then, G is an upper semicontinuous decomposition of S^3 into points and tame

arcs. Let S^3/G denote the associated decomposition space, the dogbone space of this note.

From [1], we have that S^3/G is not topologically S^3.

The methods of Casler [5] can be easily modified to show that H_2^*, the natural projection of H_2 in S^3/G, is a self-universal crumpled cube.

Since H_1^*, the natural projection of H_1 in S^3/G, is a crumpled cube and S^3/G is the result of a sewing of H_1^* and H_2^*, we have that H_2^* is not a universal crumpled cube.

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