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ON MEASURES ASSOCIATED TO SUPERHARMONIC FUNCTIONS

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ABSTRACT. Let u be a superharmonic function in an open set Ω in $\boldsymbol{R}^n$ and let μ be the positive Radon measure associated to u, i.e. μ is a negative constant multiple of the distributional Laplacian Δu of u. Using mostly elementary techniques, the paper deals with the properties of μ in the large, when $u>0$ and $\Omega=\boldsymbol{R}^n$, and in the small, in some neighbourhood of a point in Ω.

1. **Introduction and main results.** Before presenting our main results, we give three lemmas.

LEMMA 1. *If u is positive and superharmonic in $\boldsymbol{R}^n$ with $n\geqq 3$, then as $\rho\to+\infty$, the peripheral mean $\mathscr{M}(u, x, \rho)$ (of u on the sphere $S(x, \rho)$ of centre x and radius ρ) and the volume mean $\mathscr{A}(u, x, \rho)$ (in the open ball $B(x, \rho)$) have a common limit, λ, say, independent of x in $\boldsymbol{R}^n$. Further, $u\geqq\lambda$ in $\boldsymbol{R}^n$.*

Lemma 1 is known, at least implicitly. Here we give a simple proof (see §2) without using the notion of greatest harmonic minorant.

The measure μ associated to a superharmonic function u in an open set Ω in $\boldsymbol{R}^n$ with $n\geqq 2$ is a nonnegative (Radon) measure in Ω such that

$$\int_\Omega \varphi\, d\mu = -p_n\int_\Omega u(x)\, \Delta\varphi(x)\, dx \qquad (\varphi\in\mathscr{D}(\Omega)), \tag{1}$$

where $\mathscr{D}(\Omega)$ is the set of infinitely differentiable functions vanishing outside a compact set in Ω, Δ is the Laplacian operator, and $p_2=(2\pi)^{-1}$, $p_n=((n-2)s_n)^{-1}$ $(n\geqq 3)$, s_n being the surface-area of the unit sphere $S(0, 1)$ in $\boldsymbol{R}^n$. (For a nondistributional approach to μ and related topics, see, e.g., du Plessis [3].)

LEMMA 2. *If u is positive and superharmonic in $\boldsymbol{R}^n$ with $n\geqq 3$ and μ is the measure associated to u, then*

$$\mu(\bar{B}(x, \rho))\leqq C(n)\rho^{n-2}\mathscr{A}(u, x, \rho) \qquad (x\in R^n,\ \rho>0), \tag{2}$$

where $C(n)$ is a positive constant depending only on n.

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Lemma 2 follows easily from (1) (see §3). Lemmas 1 and 2 become obvious when $n=2$ since the only positive superharmonic functions in $\boldsymbol{R}^2$ are constants. However, (1) yields (see §4) a local form of Lemma 2 when $n=2$:

LEMMA 3. *If u is positive and superharmonic in the open disc $B(0, \rho_0)$ in $\boldsymbol{R}^2$, μ is the measure associated to u and $0<\theta<1$, then*

$$\mu(\bar{B}(0, \rho)) \leqq C(\theta)\mathscr{A}(u, 0, \rho) \qquad (0 < \rho \leqq \theta\rho_0), \tag{3}$$

where $C(\theta)$ is a positive constant depending only on θ.

In this paper we give three results on the behaviour of μ in the large (Theorem 1) and in the small (Theorems 2 and 3).

THEOREM 1. *If u is positive and superharmonic in $\boldsymbol{R}^n$ with $n\geqq 3$ and μ is the measure associated to u, then*

$$\lim_{\rho\to+\infty} \rho^{2-n}\mu(\bar{B}(x, \rho)) = 0 \qquad (x \in R^n). \tag{4}$$

Further, $2-n$ is the largest possible negative power of ρ for (4) to hold.

Here is the proof of (4). We may suppose that u is not constant. By Lemma 1, $w=u-\lambda>0$ in $\boldsymbol{R}^n$ and $\mathscr{A}(w, x, \rho)\to 0$ as $\rho\to+\infty$. Since μ is also associated to w, (4) follows from (2) applied to w. The counter-example to justify the second part of Theorem 1 is given in §5.

THEOREM 2. *Let u be superharmonic in an open set Ω in $\boldsymbol{R}^n$ with $n\geqq 2$ and let μ be the measure associated to u. Then*

$$u(x_0) < +\infty \Rightarrow \mu(\{x_0\}) = 0 \qquad (x_0 \in \Omega). \tag{5}$$

The converse is not true.

The proof of (5) is as follows. When $n\geqq 3$, there exist two functions in $\boldsymbol{R}^n$, v positive and superharmonic, h harmonic such that

$$u = v + h \tag{6}$$

in some neighbourhood ω of x_0 (Brelot [**2**, III, §1]). Clearly $v(x_0)<+\infty$ and μ is also associated to v in ω; (5) follows from (2) applied to v at x_0 (as $\rho\to 0$). When $n=2$, we take $v=u+\varepsilon-u(x_0)$ where $\varepsilon>0$. The function $v>0$ and superharmonic in some open disc $B(x_0, \rho_\varepsilon)$ and has μ as associated measure. By (3) applied to v at x_0,

$$\mu(\{x_0\}) \leqq \mu(\bar{B}(x_0, \tfrac{1}{2}\rho_\varepsilon)) \leqq C(\tfrac{1}{2})\mathscr{A}(v, x_0, \tfrac{1}{2}\rho_\varepsilon) \leqq C(\tfrac{1}{2})v(x_0) = \varepsilon C(\tfrac{1}{2}),$$

which implies that $\mu(\{x_0\})=0$. The last part of Theorem 2 is dealt with in §6.

THEOREM 3. *Let u be superharmonic in an open set Ω of $\boldsymbol{R}^n$ with $n \geqq 2$, $x_0 \in \Omega$ and μ the measure associated to u. Then $\mu(\{x_0\})=0$ if and only if*

$$\lim_{\rho \to 0} \rho^{n-2} \mathscr{A}(u, x_0, \rho) = 0 \qquad (n \geqq 3),$$

$$\lim_{\rho \to 0} \{1/\log(1/\rho)\} \mathscr{A}(u, x_0, \rho) = 0 \qquad (n = 2).$$

Same results hold when $\mathscr{A}$ is replaced by $\mathscr{M}$.

The fact that $\mathscr{A}$ and $\mathscr{M}$ can be interchanged follows immediately from

$$\mathscr{M}(u, x_0, \rho) \leqq \mathscr{A}(u, x_0, \rho) \leqq \mathscr{M}(u, x_0, \beta_n \rho),$$

where the constant β_n of the second inequality (due to Beardon [1]), is given by $\beta_2 = e^{-1/2}$, $\beta_n = (2/n)^{1/(n-2)}$ $(n \geqq 3)$.

Next, we note that Lemma 2 gives the 'if' part of Theorem 3 when $n \geqq 3$. In fact, by using the functions v and h in (6), we get that $\rho^{n-2}\mathscr{A}(v, x_0, \rho) \to 0$ as $\rho \to 0$, and hence $\mu(\{x_0\})=0$ by (2).

The proofs of the remaining parts of Theorem 3 (see §8 for the 'only if' part, §11 for the 'if' part when $n=2$) require two results which are more involved than those we used so far. The first one is the local Riesz decomposition theorem which we state (to simplify notations) for balls (discs when $n=2$) only.

THEOREM A. *Let Ω be open in $\boldsymbol{R}^n$ with $n \geqq 2$ and $\bar{B}(0, \rho) \subset \Omega$. If u is superharmonic in Ω and μ is associated to u, then there is a harmonic function h_ρ in $B(0, \rho)$ such that*

$$u(x) = \int_{B(0,\rho)} \sigma(|x-y|)\, d\mu(y) + h_\rho(x) \qquad (|x| < \rho), \tag{7}$$

where

$$\sigma(t) = \log(1/t) \quad (n = 2), \qquad \sigma(t) = t^{2-n} \quad (n \geqq 3). \tag{8}$$

The second result (proved in §7) is a simple application of Theorem A:

LEMMA 4. *Under the hypotheses of Theorem* A,

$$\mathscr{M}(u, 0, \rho) = \sigma(\rho)\mu(B(0, \rho)) + h_\rho(0). \tag{9}$$

Once that (7) and (9) are given, it is worthwhile noting that they yield two more theorems in the light of our previous results. Without using the notion of greatest harmonic minorant, we have

THEOREM B (RIESZ DECOMPOSITION IN $\boldsymbol{R}^n$). *If $u > 0$ and superharmonic in $\boldsymbol{R}^n$ with $n \geqq 3$, μ is associated to u and λ is the constant in Lemma* 1, *then*

$$u(x) = \lambda + \int_{\boldsymbol{R}^n} |x-y|^{2-n}\, d\mu(y) \qquad (x \in \boldsymbol{R}^n). \tag{10}$$

In fact, it is enough to prove (10) at $x=0$. Applying Lemma 1 and (4) (both at $x=0$) to (9) we get that

$$\lim_{\rho\to+\infty} h_\rho(0) = \lambda, \tag{11}$$

which, together with (7) at $x=0$, yields (10).

The second theorem sharpens Lemmas 2 and 3.

THEOREM 4. *The inequalities* (2) *and* (3) *can be replaced by*

$$\mu(\bar{B}(x, \rho)) \leqq \rho^{n-2}\mathscr{M}(u, x, \rho) \qquad (x \in \boldsymbol{R}^n,\ \rho > 0), \tag{12}$$

$$\mu(\bar{B}(0, \rho)) \leqq \{1/\log(1/\theta)\}\mathscr{M}(u, 0, \rho) \qquad (0 < \rho \leqq \theta\rho_0) \tag{13}$$

respectively. Further, (12) *and* (13) *are sharp.*

Inequalities (12) and (13) are proved in §§9 and 10. To show that they are sharp, it is enough to consider the fundamental superharmonic function $\sigma(|x|)$ (see (8)). The measure associated to σ is, by the choice of p_n in (1), the Dirac measure δ_0 at the origin 0 (see, e.g., Brelot [**2**, IV, §2]). Hence (12) becomes an equality at $x=0$ for all positive ρ. When $n=2$, $\sigma>0$ in $B(0, 1)$ and (13) becomes an equality when $\rho=\theta<1=\rho_0$.

2. **Proof of Lemma 1.** Since $\mathscr{M}(u, x, \rho)$ is positive, real-valued and decreasing, it has a nonnegative finite limit, $\lambda(x)$, say, as $\rho\to+\infty$. Further

$$\mathscr{A}(u, x, \rho) = n\rho^{-n}\int_0^\rho \mathscr{M}(u, x, t)t^{n-1}\,dt$$

and an easy technique (consisting in writing the integral on the right as the sum of two integrals on $(0, \rho_0)$ and (ρ_0, ρ)) gives that $\mathscr{A}(u, x, \rho)\to\lambda(x)$ as $\rho\to+\infty$. To complete the proof, it is enough to show that $\lambda(x)\geqq\lambda(y)$ for any two points x, y in $\boldsymbol{R}^n$, and this follows from the inequality (as $\rho\to+\infty$)

$$\mathscr{A}(u, x, \rho + |x-y|) \geqq \{\rho/(\rho + |x-y|)\}^n\mathscr{A}(u, y, \rho) \qquad (\rho > 0).$$

3. To prove Lemma 2, we choose a function φ_0 in $\mathscr{D}(\boldsymbol{R}^n)$ such that $\varphi_0\geqq0$, $\varphi_0(x)=1$ $(|x|\leqq1)$, $\varphi_0(x)=0$ $(|x|\geqq2)$. Let M_0 be the supremum of $|\Delta\varphi_0|$. We apply (1) to the function

$$\varphi(x) = \varphi_0(x/\rho) \qquad (x \in R^n).$$

Clearly φ is a nonnegative function in $\mathscr{D}(\boldsymbol{R}^n)$, equals 1 on $\bar{B}(0, \rho)$, vanishes outside $B(0, 2\rho)$, and $|\Delta\varphi|\leqq M_0/\rho^2$. Hence

$$\begin{aligned}\mu(\bar{B}(0, \rho)) &= \int_{\bar{B}(0,\rho)}\varphi\,d\mu \leqq \int_{R^n}\varphi\,d\mu = -p_n\int_{B(0,2\rho)}u(x)\Delta\varphi(x)\,dx\\ &\leqq p_n\rho^{-2}M_0\int_{B(0,2\rho)}u(x)\,dx = p_n v_n 2^n M_0\rho^{n-2}\mathscr{A}(u, 0, 2\rho)\end{aligned}$$

where v_n is the volume of $B(0, 1)$. The decreasing property of $\mathscr{A}$ gives (2) at $x=0$, and hence everywhere in $\boldsymbol{R}^n$.

4. To prove Lemma 3, we choose φ_0 in $\mathscr{D}(\boldsymbol{R}^2)$ such that $\varphi_0 \geqq 0$, $\varphi_0(x)=1$ $(|x| \leqq \theta)$, $\varphi_0(x)=0$ $(|x| \geqq \frac{1}{2}(\theta+1))$. With $M_\theta = \sup|\Delta\varphi_0|$ and $\varphi(x)=\varphi_0(\theta x/\rho)$, we obtain that φ is a nonnegative function in $\mathscr{D}(\boldsymbol{R}^2)$, equals 1 on $\bar{B}(0, \rho)$, vanishes outside $B(0, \frac{1}{2}\rho(\theta+1)/\theta)$ (so that the support of φ is in $B(0, \rho_0)$), and that $|\Delta\varphi| \leqq \theta^2\rho^{-2}M_\theta$. Steps identical to those in §3 (with $B(0, 2\rho)$ replaced by $B(0, \frac{1}{2}\rho(\theta+1)/\theta)$) yield

$$\begin{aligned} \mu(\bar{B}(0, \rho)) &\leqq \tfrac{1}{8}(\theta + 1)^2 M_\theta \mathscr{A}(u, 0, \tfrac{1}{2}\rho(\theta + 1)/\theta) \\ &\leqq \tfrac{1}{8}(\theta + 1)^2 M_\theta \mathscr{A}(u, 0, \rho) \end{aligned}$$

whenever $0 < \rho \leqq \theta\rho_0$.

5. To prove the second part of Theorem 1, we show that if $0 < \alpha < n-2$, then

$$\rho^{2-n+\alpha}\mu(\bar{B}(0, \rho)) \tag{14}$$

does not necessarily tend to 0 as $\rho \to +\infty$. Let u be the function given by

$$u(0) = +\infty, \qquad u(x) = |x|^{-\alpha} \quad (x \neq 0). \tag{15}$$

The function $u>0$ and superharmonic in $\boldsymbol{R}^n$ and, by a simple computation,

$$\mathscr{A}(u, 0, \rho) = n(n - \alpha)^{-1}\rho^{-\alpha}, \quad \Delta u(x) = -\alpha(n - 2 - \alpha)\,|x|^{-\alpha-2} \quad (x \neq 0). \tag{16}$$

The first equality and (2) at $x=0$ imply that

$$0 = \lim_{\rho \to 0} \mu(\bar{B}(0, \rho)) = \mu(\{0\}). \tag{17}$$

Since outside the origin μ equals $-p_n\Delta u$ times the Lebesgue measure, we obtain by (17) and (16),

$$\mu(\bar{B}(0, \rho)) = \lim_{\delta \to 0} \int_{\delta \leqq |x| \leqq \rho} -p_n \Delta u(x)\, dx = \alpha(n - 2)^{-1}\rho^{n-2-\alpha}.$$

Hence, with the function u of (15), (14) does not tend to 0 as $\rho \to +\infty$.

6. We now show that the converse of (5) in Theorem 2 is not true. When $n \geqq 3$, (15) and (17) provide the counterexample since $u(0) = +\infty$ and $\mu(\{0\})=0$. When $n=2$, consider the function u in $\boldsymbol{R}^2$ given by

$$u(x) = u(x_1, x_2) = \int_0^{1/e} -\tfrac{1}{2} \log\{(x_1 - t)^2 + x_2^2\} \frac{dt}{t(\log t)^2}. \tag{18}$$

With the measure ν defined by

$$\nu(f) = \int_0^{1/e} f(t, 0) \frac{dt}{t(\log t)^2} \tag{19}$$

for any f continuous with compact support in $\mathbf{R}^2$, we have

$$u(x) = \int_{R^2} \sigma(|x-y|)\, d\mu(y) \qquad (x \in R^2).$$

Formula (19) shows that ν is a nonnegative Radon measure with compact support and hence u is superharmonic in $\mathbf{R}^2$ (see, e.g., Brelot [**2**, IV, §1]). Further ν is the measure μ associated to u. In fact (without using distributional convolutions) the measure associated to $\sigma(|x-y|)$ (as a function of x) is the Dirac measure δ_y at y and hence, by two applications of (1) and Fubini's theorem,

$$2\pi\mu(\varphi) = -\int_{R^2}\int_{R^2} \sigma(|x-y|)\Delta\varphi(x)\, dx\, d\nu(y) = 2\pi \int_{R^2} \varphi(y)\, d\nu(y)$$

for all φ in $\mathscr{D}(\mathbf{R}^2)$; this, in turn, implies that $\mu=\nu$ (see, e.g., du Plessis [**3**, Theorem 1.30]). Finally (18) and (19) give easily the required conditions $u(0)=+\infty$ and $\mu(\{0\})=0$ respectively.

7. To prove Lemma 4 let r be such that $0<r<\rho$. By (7),

$$\mathscr{M}(u, 0, r) = (s_n r^{n-1})^{-1} \int_{S(0,r)} \int_{B(0,\rho)} \sigma(|x-y|)\, d\mu(y)\, ds(x) + h_\rho(0)$$

where ds is the surface-area (arc length when $n=2$) element on $S(0, r)$. With y fixed in $B(0, \rho)$ it is well known that the function σ_y, given by $\sigma_y(x)=\sigma(|x-y|)$, satisfies

$$\begin{aligned} \mathscr{M}(\sigma_y, 0, r) &= \sigma(r) \qquad (r \geqq |y|), \\ &= \sigma(|y|) \qquad (r \leqq |y|), \end{aligned}$$

and hence by Fubini's theorem (which holds even when $n=2$)

$$\begin{aligned} \mathscr{M}(u, 0, r) - h_\rho(0) &= \int_{B(0,\rho)} \mathscr{M}(\sigma_y, 0, r)\, d\mu(y) \\ &= \sigma(r)\mu(\bar{B}(0, r)) + \int_{r<|y|<\rho} \sigma(|y|)\, d\mu(y). \end{aligned}$$

Lemma 4 follows from this last equality as $r \to \rho -$.

8. Here we prove the 'only if' part of Theorem 3 ($n \geqq 2$). We take the origin 0 at x_0. We may suppose that $u>0$ in some $\bar{B}(0, \rho_0)$ in Ω (since the

addition of a constant c does not alter μ and the required limit property holds for u if and only if it holds for $u+c$). Let ε be a positive number. By hypothesis $\mu(\{0\})=0$ and hence $\mu(B(0, \rho))\to 0$ as $\rho\to 0$. We choose two numbers r, ρ such that $0<r<\rho\leqq\min\{1, \rho_0\}$, $\mu(B(0, \rho))\leqq\frac{1}{2}\varepsilon$, $h_\rho(0)/\sigma(r)\leqq\frac{1}{2}\varepsilon$. Theorem A for radii r and ρ yields

$$\int_{B(0,r)} \sigma(|x-y|)\,d\mu(y) + h_r(x) = \int_{B(0,\rho)} \sigma(|x-y|)\,d\mu(y) + h_\rho(x)$$

whenever $|x|<r$. Since the integral on the left is (superharmonic whence) finite q.p., the equality

$$h_r(x) = \int_{r\leqq|y|<\rho} \sigma(|x-y|)\,d\mu(y) + h_\rho(x)$$

holds q.p. and hence everywhere in $B(0, r)$, in particular at $x=0$. This last case and Lemma 4 (with radius r) imply that

$$\begin{aligned} 0 \leqq \mathscr{M}(u, 0, r)/\sigma(r) &= \mu(B(0, r)) + \int_{r\leqq|y|<\rho} \sigma(|y|)/\sigma(r)\,d\mu(y) + \tfrac{1}{2}\varepsilon \\ &\leqq \mu(B(0, r)) + \int_{r\leqq|y|<\rho} d\mu(y) + \tfrac{1}{2}\varepsilon \\ &= \mu(B(0, \rho)) + \tfrac{1}{2}\varepsilon \leqq \varepsilon. \end{aligned}$$

(The hypothesis $r<\rho\leqq 1$ is used in the first two inequalities when $n=2$.) Since ε is arbitrary, we obtain the required result.

9. To prove (12) of Theorem 4, it is enough to work at $x=0$ and apply previous results. Since $\int_{B(0,\rho)}|y|^{2-n}\,d\mu(y)$ is clearly an increasing function of ρ, the formula (7) at $x=0$ gives that $h_\rho(0)$ is decreasing. Hence $h_\rho(0)\geqq\lambda\geqq 0$ by (11) and Lemma 1, and (12) follows from (9).

10. We now prove (13). Suppose that $\theta<c<1$ and let $\rho_1=c\rho_0$. Clearly $\bar{B}(0, \rho_1)\subset B(0, \rho_0)$ and, by Theorem A at $x=0$, we have, for any $\rho\leqq\rho_1$,

$$u(0) = \int_{B(0,\rho)} \log(\rho_1/|y|)\,d\mu(y) + g(\rho) \tag{20}$$

where

$$g(\rho) = h_\rho(0) + \log(1/\rho_1)\mu(B(0, \rho)).$$

The integral in (20) is an increasing function of ρ in $(0, \rho_1]$ and hence g is decreasing in $(0, \rho_1]$. Since further $g(\rho_1)$ is, by (9), equal to $\mathscr{M}(u, 0, \rho_1)$ which is positive, we obtain that g is positive in $(0, \rho_1]$. If now $\rho\leqq\theta\rho_0$ we have, by Lemma 4,

$$\begin{aligned} \mathscr{M}(u, 0, \rho) &= g(\rho) + \log(\rho_1/\rho)\mu(B(0, \rho)) \\ &> \log(\rho_1/\rho)\mu(B(0, \rho)) \geqq \log(c/\theta)\mu(B(0, \rho)). \end{aligned}$$

Thus

$$\mu(B(0,\rho)) < \{1/\log(c/\theta)\}\mathscr{M}(u,0,\rho) \qquad (\rho \leqq \theta\rho_0)$$

for any c such that $\theta<c<1$. As $c\to 1-$, we get

$$\mu(B(0,\rho)) \leqq \{1/\log(1/\theta)\}\mathscr{M}(u,0,\rho) \qquad (\rho \leqq \theta\rho_0).$$

If in this last inequality we replace ρ and θ by $\rho+\varepsilon\rho_0$ and $\theta+\varepsilon$ respectively, we get

$$\mu(\bar{B}(0,\rho)) \leqq \mu(B(0,\rho+\varepsilon\rho_0)) \leqq \{1/\log(1/(\theta+\varepsilon))\}\mathscr{M}(u,0,\rho+\varepsilon\rho_0)$$

which in turn gives (13) as $\varepsilon\to 0+$.

11. We finally prove the 'if' part of Theorem 3 (at $x_0=0$) in the case where $n=2$. We may suppose that $u>0$ in some $\bar{B}(0,\rho_0)$ in Ω (see §8). Since (13) holds when $\rho=\theta\rho_0$ we can replace θ by ρ/ρ_0 and hence

$$\mu(\bar{B}(0,\rho)) \leqq \{1/\log(\rho_0/\rho)\}\mathscr{M}(u,0,\rho) \qquad (\rho<\rho_0).$$

Since the hypothesis of the theorem implies that the right-hand side tends to zero as $\rho\to 0$, the result follows.

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