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THE COMPLETION OF A RING WITH A VALUATION

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ABSTRACT. This paper proves three main results: the completion of a commutative ring with respect to a Manis valuation is an integral domain; a necessary and sufficient condition is given that the completion be a field; and the completion is a field when the valuation is Harrison and the value group is archimedean ordered.

1. **Introduction.** In [**1**] we defined a generalized pseudovaluation on a ring R and found explicitly the completion of R with respect to the topology induced on R by such a pseudovaluation. Using this inverse limit characterization of the completion of R we show, in §2, that when the codomain of a valuation on R is totally ordered, the completion of R with respect to the valuation has no (nonzero) divisors of zero and the valuation on R can be extended to a valuation on the completion. In §3, R is assumed to have an identity and so, from [**1**, §2], the completion has an identity: a necessary and sufficient condition is given such that each element of the completion of R has a right inverse.

§§2 and 3 are immediately applicable to a Manis valuation φ on a commutative ring R with identity [**3**], where the valuation φ is surjective and the set $\varphi(R)$ is a totally ordered group. A Harrison valuation on R is a Manis valuation φ on R such that the set $\{x : x \in R,\ \varphi(x) > \varphi(1)\}$ is a finite Harrison prime [**2**]. In §4 we show that the completion of a ring with respect to a Harrison valuation is a field if the value group is archimedean ordered.

Let R be a ring (not necessarily either commutative or with identity). Let S be a totally ordered quasi-residuated [**1**, §1] semigroup and let S_0 be the disjoint union of S and a zero element 0_S with the properties: $0_S 0_S = 0_S$; and, for any $s \in S$, $0_S > s$ and $s\, 0_S = 0_S = 0_S\, s$. Let $\varphi : R \to S_0$ be a valuation on R. That is, for all $a, b \in R$,

(1.1) $\varphi(ab) = \varphi(a)\varphi(b)$;

(1.2) $\varphi(a-b) \geqq \min\{\varphi(a), \varphi(b)\}$;

(1.3) $\varphi(0) = 0_S$;

(1.4) the set $\varphi(R) \backslash \{0_S\}$ is nonempty.

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Note that, if $\varphi(a)\neq\varphi(b)$, then

$$\varphi(a-b)=\min\{\varphi(a),\varphi(b)\}.$$

Now if S has a greatest element π, it is clear from [**1**] that the completion of R with respect to φ is isomorphic to the ring $R/\{x:x\in R,\ \varphi(x)\geqq\pi\}$. Hence let S have no greatest element.

Following [**1**], let $\{P_s\}_{s\in S}$ be the generalized filtration defined on R as follows: For each $s\in S$, $P_s=\{x:x\in R,\ \varphi(x)\geqq s\}$. Note that, in this case, $\bigcup_{s\in S}P_s=R$. From [**1**, §2], the completion of R with respect to φ is the topological ring $\boldsymbol{R}=\operatorname{proj\,lim} R/P_s$, with zero $\mathbf{0}=\{P_s\}_{s\in S}$, and addition, multiplication and topology defined in [**1**, §2]. For reference, $\boldsymbol{R}=\{\{\xi_s\}_{s\in S}\in\prod_{s\in S}R/P_s$: for all $s,\ t\in S$ such that $s\geqq t,\ \xi_s\subseteq\xi_t\}$. When there is no risk of ambiguity, $\{\xi_s\}_{s\in S}$ will be written as $\{\xi_s\}$.

2. ***R* has no divisors of zero.** First we need

LEMMA 1. *Let $\{\xi_s\}_{s\in S}\in\boldsymbol{R}\backslash\{\mathbf{0}\}$. Then there exist $u,\ v\in S$ with $u<v$ such that $\xi_u=P_u$, $\xi_v\neq P_v$, and, for all $x,\ y\in\xi_v$, $u\leqq\varphi(x)=\varphi(y)<v$.*

PROOF. (i) Let $s\in S$, $x\in\xi_s$. Since $\bigcup_{s\in S}P_s=R$, there exists $t\in S$ such that $x\in P_t$. If $t\geqq s$, then $x\in P_s$ from [**1**, (1.2)], and so $\xi_s=P_s$. Otherwise $t<s$, $\xi_s\subseteq\xi_t$, and so $\xi_t=x+P_t=P_t$. Hence there exists $u\in S$ such that $\xi_u=P_u$.

(ii) Since $\{\xi_s\}_{s\in S}\neq\mathbf{0}$, there exists $v\in S$ such that $\xi_v\neq P_v$. Now if $v\leqq u$, then $P_u=\xi_u\subseteq\xi_v$, and so $\xi_v=P_v$. Therefore $v>u$.

(iii) Now $\varphi(x)<v$ for all $x\in\xi_v$, since otherwise $\xi_v=P_v$. Since $v>u$, $\xi_v\subseteq\xi_u=P_u$ and so $\varphi(x)\geqq u$ for all $x\in\xi_v$. Suppose there exist $y,\ z\in\xi_v$ such that $\varphi(y)\neq\varphi(z)$. Then $\min\{\varphi(y),\ \varphi(z)\}=\varphi(y-z)\geqq v$ since $y-z\in P_v$, which contradicts $\varphi(y),\ \varphi(z)<v$. Hence $\varphi(x)=\varphi(y)$ for all $x,\ y\in\xi_v$. This completes the proof.

THEOREM 1. *$\boldsymbol{R}$ has no divisors of zero.*

PROOF. Let $\{\xi_s\},\ \{\eta_s\}\in\boldsymbol{R}$ such that $\{\xi_s\}\{\eta_s\}=\mathbf{0}$. Suppose that $\{\xi_s\}$, $\{\eta_s\}\neq\mathbf{0}$. Then, by Lemma 1, there exist $a,\ b,\ t,\ u\in S$ such that, for all $x\in\xi_t$, $\varphi(x)=a$; and, for all $y\in\eta_u$, $\varphi(y)=b$.

Since S has no maximal element, there exists $c\in S$ such that $c>ab$; and, by the definition of multiplication in $\boldsymbol{R}$ [**1**, §2], there exists $v\in S$ such that, for all $x\in\xi_v$ and for all $y\in\eta_v$, $xy\in P_c$.

Let $w\geqq t,\ u,\ v$. Then, for all $x\in\xi_w\subseteq\xi_t$ and for all $y\in\eta_w\subseteq\eta_u$, $\varphi(xy)=ab<c$, which contradicts $xy\in P_c$.

Hence $\{\xi_s\}=\mathbf{0}$ or $\{\eta_s\}=\mathbf{0}$ which completes the proof.

COROLLARY. *The completion of a commutative ring with respect to a nontrivial Manis valuation is an integral domain.*

Lemma 1 also enables us to define a valuation on $\boldsymbol{R}$. Let $\{\xi_s\} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}$. Then there exists $t \in S$ such that $\xi_t \neq P_t$. Define $\boldsymbol{\varphi}(\{\xi_s\}) = \varphi(x)$ where $x \in \xi_t$. Define $\boldsymbol{\varphi}(\boldsymbol{0}) = 0_S$. Then it is a straightforward task to show that $\boldsymbol{\varphi}: \boldsymbol{R} \to S_0$ is a well defined valuation on $\boldsymbol{R}$. Moreover, the valuation $\boldsymbol{\varphi}$ on $\boldsymbol{R}$ is an "extension" of the valuation φ on R in the sense that $\boldsymbol{\varphi}$ restricted to the image $i(R)$ of R in $\boldsymbol{R}$, defined in [**1**, Theorem 2.1], satisfies $\boldsymbol{\varphi}(i(x)) = \varphi(x)$ for each $x \in R$. And it is easy to show that $\boldsymbol{\varphi}$ defines the topology $\mathscr{T}$ [**1**, §2] on $\boldsymbol{R}$.

3. **When elements of the completion have inverses.** Let R have identity 1. Then, from [**1**, Proposition 2.1], $\boldsymbol{R}$ has identity $\{1+P_s\}_{s \in S}$.

PROPOSITION 1. *Each element of $\boldsymbol{R}$ has a right inverse if and only if* (1) *given $a \in R$ such that $\varphi(a) \neq 0_S$, and given $s \in S$, there exists $b \in R$ such that $\varphi(ab-1) \geqq s$.*

PROOF. (i) Let each element of $\boldsymbol{R}$ have a right inverse. Let $a \in R$ be such that $\varphi(a) \neq 0_S$, and let $w \in S$. Clearly, $\{a+P_s\} \neq \boldsymbol{0}$ since S has no maximal element. Hence there exists $\{\eta_s\} \in \boldsymbol{R}$ such that $\{a+P_s\}\{\eta_s\} = \{1+P_s\}$. By the definition of multiplication in $\boldsymbol{R}$, there exists $t \in S$ such that, for all $x \in \eta_t$, $ax \in 1+P_w$. Hence (1) holds.

(ii) Conversely, let (1) hold. Denote $\varphi(1)$ by 1_S. Let $\{\xi_s\} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}$. By Lemma 1 there exist $\rho, \sigma \in S$ such that $\xi_\rho \neq P_\rho$ and, for all $x \in \xi_\rho$, $\varphi(x) = \sigma$; σ is a constant for the chosen $\{\xi_s\}$.

For each $s \in S$, let $x_s \in \xi_s$ and let $y_s \in R$ be such that $\varphi(x_s y_s - 1) > \sigma s$ (S has no maximal element); let $m_s \in S$ be such that $m_s > \sigma s$ and let $g(s) \in S$ be such that $g(s) \geqq m_s \sigma, s$. Let $t \in S$ be such that $\sigma t > 1_S$ and $t \geqq \rho$. Then, for each $s \in S$ such that $s \geqq t$, $\sigma\varphi(y_s) = \varphi(x_s y_s) = 1_S$. Define, for each $s \in S$,

$$\begin{aligned} \eta_s &= y_{g(s)} + P_s \quad \text{if } s \geqq t, \\ &= y_{g(t)} + P_s \quad \text{if } s < t. \end{aligned}$$

We shall show that $\{\xi_s\}\{\eta_s\} = \{1+P_s\}$.

(a) We must show that, for each $s \in S$, η_s is well defined. Accordingly, for each $s \in S$, let $w_s \in \xi_s$, $z_s \in R$ and $\tau, u, n_s, h(s) \in S$ be such that $\xi_\tau \neq P_\tau$; $\varphi(w_s z_s - 1) > \sigma s$; $n_s > \sigma s$ and $h(s) \geqq n_s \sigma, s$; $\sigma u > 1_S$ and $u \geqq \tau$; and

$$\begin{aligned} \zeta_s &= z_{h(s)} + P_s \quad \text{if } s \geqq u, \\ &= z_{h(u)} + P_s \quad \text{if } s < u. \end{aligned}$$

We must show that, for each $s \in S$, $\zeta_s = \eta_s$.

CLAIM. Let $a, b, c \in S$ be such that $a, b \geqq c \geqq t$. Then $z_{h(a)} - y_{g(b)} \in P_c$.

SUBPROOF. We shall refer to $h(a)$ and $g(b)$ as h and g respectively. Now

$$w_h(z_h - y_g) = (w_h z_h - x_g y_g) + (x_g - w_h)y_g.$$

Clearly, $\varphi\{w_h z_h - 1 - (x_g y_g - 1)\} > \sigma c$. Also $\varphi(x_g - w_h) \geqq r\sigma$ where $r = \min\{m_b, n_a\} > \sigma c$. Hence $\varphi\{(x_g - w_h)y_g\} > \sigma c$ since $\sigma\varphi(y_g) = 1_S$, and so $\varphi\{w_h(z_h - y_g)\} > \sigma c$. Thus $\varphi(z_h - y_g) \geqq c$ since $\varphi(w_h) = \sigma$. This proves the claim.

Let $s \in S$. Without loss of generality, let $u \geqq t$. Then, by applying the claim to each of the cases $s \geqq u$, $u \geqq s \geqq t$, $t > s$, we see that $\zeta_s = \eta_s$. Thus η_s is well defined for each $s \in S$.

(b) By further application of the claim, it is easy to show that $\{\eta_s\}_{s \in S} \in \boldsymbol{R}$.

(c) Finally we must show that $\{\xi_s\}\{\eta_s\} = \{1 + P_s\}$. Let $\{\xi_s\}\{\eta_s\} = \{\Omega_s\}$. Let $s \in S$. Then there exists $v \in S$ such that $xy \in \Omega_s$ for all $x \in \xi_v$, $y \in \eta_v$. Let $d \in S$ be such that $\sigma d \geqq s$ and $d \geqq v, t$. Then $x_{g(d)} \in \xi_v$, $y_{g(d)} \in \eta_v$ and $\varphi(x_{g(d)} y_{g(d)} - 1) \geqq s$. Hence $\Omega_s = x_{g(d)} y_{g(d)} + P_s = 1 + P_s$ which completes the proof of the proposition.

COROLLARY. *Let $\varphi: R \to S_0$ be a nontrivial Manis valuation on a commutative ring R with identity* 1. *Then the completion of R with respect to φ is a field if and only if condition* (1) *holds.*

4. **Harrison valuations.** In [**2**, pp. 11 and 12 and Proposition 2.2, p. 14], Harrison shows that, given a finite prime P of a commutative ring R with identity, one can define a Manis valuation $\varphi_P: R \to \Gamma_P$ on R such that the set $\{x : x \in R,\ \varphi_P(x) > \varphi_P(1)\} = P$. Using the corollary to Proposition 1, we have:

THEOREM 2. *Let $\varphi: R \to S_0$ be a nontrivial Harrison valuation on a commutative ring R with identity 1_R, such that S be archimedean ordered. Then the completion of R with respect to φ is a field.*

PROOF. Let $a \in R$ be such that $\varphi(a) \neq 0_S$ and let $s \in S$. Now $\{x : x \in R,\ \varphi(x) > 1_S\} = P$, a finite Harrison prime of R. Suppose $aR \subseteq P$. Then, for all $x \in R$, $\varphi(a)\varphi(x) > 1_S$, which contradicts $\varphi(a) \neq 0_S$ since φ is surjective and S is a totally ordered group. Hence $aR + P$ properly contains P. But $aR + P$ is closed under addition and multiplication. Thus $-1_R \in aR + P$, since P is a Harrison prime: that is, there exist $c \in R$, $d \in P$ such that $ac - 1_R = d$.

Since $\varphi(d) > 1_S$ and S is archimedean ordered, there exists a natural number m such that $\varphi(d^m) = \{\varphi(d)\}^m \geqq s$. Let n be an odd integer such that $n \geqq m$. Then $\varphi(d^n) \geqq s$. Now $d^n = (ac - 1_R)^n = ab - 1_R$ where $b \in R$. The theorem then follows from Proposition 1.

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