PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, Number 1, November 1972

NORMALITY OF POWERS IMPLIES COMPACTNESS

S. P. FRANKLIN AND R. C. WALKER

ABSTRACT. In this note we give a short proof for the theorem of N. Noble which asserts that *each power of a T_1-space being normal implies that the space is compact.*

Recently N. Noble proved the following result:

1. (NOBLE [N].) *If each power of a T_1-space is normal, then the space is compact.*

This remarkable theorem has been applied by Herrlich and Strecker [**H-S**] and by Franklin, Lutzer and Thomas [**F-L-T**] to give two different categorical characterizations of the category of compact Hausdorff spaces as a subcategory of the Hausdorff spaces.

Noble's original proof derived this theorem as a corollary of a more complicated one which in turn depended on a long chain of previous results of Noble and others. Keesling has given a short and very elegant derivation of Noble's theorem from two well-known theorems, one of Stone (see §2 below) and one of Morita [**K**].

Our purpose in this note is to present another short[1] proof of Noble's theorem which may point the way to an elementary proof, i.e., one which proceeds directly from the definitions without relying on any powerful theorem along the way. Our nonelementary proof relies on theorems of Stone and Glicksberg:

2. (STONE [S].) *If a product of T_1-spaces is normal, all but at most a countable number of the factor spaces must be countably compact.*

3. (GLICKSBERG [G].) *If X is completely regular, then the identity map on any pseudocompact power X^α extends to a homeomorphism from $\beta(X^\alpha)$ to $(\beta X)^\alpha$.*

PROOF OF 1. Let X^m be an uncountable power of X. Since $(X^m)^m = X^m$, X^m must be countably compact by 2 and is therefore pseudocompact. We

Received by the editors December 28, 1971.

AMS 1970 *subject classifications*. Primary 54B10, 54D30; Secondary 54D15, 54D35.

Key words and phrases. Normal, compact, product, z-ultrafilter, pseudocompact, countably compact, Stone-Čech compactification.

[1] Shortened even further by a clever observation of the referee.

can now apply 3 to obtain $\beta(X^m)=(\beta X)^m$. Now if X is not compact, there is a point p in $\beta X\backslash X$. Following **[G-J]**, we identify p with the unique free z-ultrafilter on X which converges to p. Consider the product X^p where the z-ultrafilter p is the index set. The proof is completed by showing that the assumption that p is free will provide disjoint closed sets of X^p which cannot be separated. Write Δ for the diagonal of X^p, i.e. the set of all constant functions. Write C for the set of choice functions, i.e., $C=\{x\in X^p : x_Z\in Z \text{ for } Z\in p\}$. The sets Δ and C are closed, and are disjoint since p is free. Since X^p is normal, there is a Urysohn function $f:X^p\to I$ completely separating Δ and C. But f cannot extend to $\beta(X^p)$ since the function in $(\beta X)^p=\beta(X^p)$ which is constantly p belongs to the closure in $(\beta X)^p$ of both Δ and C. Thus it must be that $\Delta\cap C\neq\varnothing$, i.e., p is fixed.

An elementary proof would result if one could show *directly* that Δ and C cannot be separated in X^p.

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Department of Mathematics, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213

Current address (S. P. Franklin): Department of Mathematics, Memphis State University, Memphis, Tennessee 38111

Current address (R. C. Walker): Department of Mathematics, Ohio Northern University, Ada, Ohio 45810