A NEW PROOF OF A THEOREM ON QUASITRIANGULAR OPERATORS

GLENN R. LUECKE

ABSTRACT. P. R. Halmos has given a proof of the equivalence of two definitions for quasitriangular operators. A short, elementary proof of this fact is given here.

In his paper Quasitriangular operators [1], Halmos proved the equivalence of the conditions (Δ₀) and (Δ₂) for operators A on Hilbert space H (dim H = \( \infty \)). An operator satisfying (Δ₀) or (Δ₂) is called quasitriangular. The proof that (Δ₀) implies (Δ₂) is trivial. However, Halmos uses a three page proof to show that (Δ₂) implies (Δ₀). The following is a short and completely elementary proof of this fact.

Operator A satisfies condition (Δ₂) if there exists a sequence \( \{E_n\} \) of (orthogonal) projections of finite rank such that \( E_n \rightarrow I \) (strong topology) and \( \|AE_n - E_nAE_n\| \rightarrow 0 \). Operator A satisfies condition (Δ₀) if for every projection P of finite rank and for every \( \varepsilon > 0 \) there exists a finite rank projection \( E \geq P \) such that \( \|AE - EAE\| < \varepsilon \).

**Theorem (Halmos).** If A satisfies condition (Δ₂), then A satisfies condition (Δ₀).

**Proof.** Use the notation above and let \( Q_n \) be the projection on \( E_n(N) \), where \( N = P(H) \). Let \( \frac{1}{2} > \delta > 0 \) be given. Since \( \dim N < \infty \) and since \( E_ng \rightarrow g \) for each \( g \in H \), there exists \( n_0 \) such that for all \( n \geq n_0 \), \( \|E_ng - g\| < \delta \|g\| \) for all \( g \in N \). Let \( n \geq n_0 \) and let \( f \in E_n(N) \), \( \|f\| = 1 \), \( f = E_ng \), \( g \in N \). Then \( \|g\| \leq \|g - E_ng\| + \|E_ng\| \leq \delta \|g\| + 1 \) so that \( \|g\| \leq (1 - \delta)^{-1} \). Then
\[
\|Q_nf - Pf\| = \|f - Pf\| = \|E_ng - PE_ng\|
\leq \|E_ng - Pg\| + \|Pg - PE_ng\|
\leq \|E_ng - g\| + \|P\| \|g - E_ng\|
\leq (1 + \|P\| \cdot \delta \|g\| \leq 2(1 - \delta)^{-1}\delta < 4\delta.
\]

Furthermore if \( f \in N \), \( \|f\| = 1 \), then \( Pf = f \) and \( Q_nf = E_nf \). Hence for \( n \geq n_0 \), \( \|Q_nf - Pf\| = \|E_nf - f\| < \delta \). Combining this with the previous

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statement we have that, for each \( n \geq n_0 \), \( \|Q_n f - Pf\| < 4\delta \) for all \( f \in N \cup E_n(N), \|f\| = 1 \). If \( f \in (N \cup E_n(N))^\perp \subseteq N^\perp \cap (E_n(N))^\perp \), then \( Q_n f = Pf = 0 \).

Taking the supremum (for each fixed \( n \geq n_0 \)) of \( \|Q_n f - Pf\| \) over all \( \|f\| = 1 \), we obtain \( \|Q_n - P\| \leq 4\delta \). Thus \( \|Q_n - P\| \to 0 \).

Define \( O_n \) so that \( E_n(H) = E_n(N) \oplus O_n \) and let \( P_n \) be the projection on \( N \oplus O_n \). Then \( P_n \) has finite rank, \( P_n \geq P \) and, since \( Q_n \) is the projection on \( E_n(N) \), \( \|E_n - P_n\| = \|Q_n - P\| \to 0 \). Thus since \( \|AE_n - E_n AE_n\| \to 0 \) and \( \|E_n - P_n\| \to 0 \), we obtain \( \|AP_n - P_n AP_n\| \to 0 \). Therefore condition \((\Delta_0)\) holds.

Reference


Department of Mathematics, Iowa State University, Ames, Iowa 50010