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SOME HOMOLOGICAL RESULTS ON CERTAIN FINITE RING EXTENSIONS

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ABSTRACT. All rings are commutative with identity and all modules are unitary. A ring R is connected if 0 and 1 are the only idempotent elements of R. R is semiconnected if the number of idempotents in R is finite.

PROPOSITION. *Suppose that R is connected, that I is a principal ideal of $R[x]$, and that $R[x]/I$ is a finitely generated R-module. Then $R[x]/I$ is a free R-module.*

PROPOSITION. *Suppose that R is semiconnected, that I is a principal ideal of $R[x]$, and that $R[x]/I$ is a finitely generated R-module. Then $R[x]/I$ is a projective R-module.*

These results are applied to integral extensions.

Introduction. Let R be a commutative ring with identity, let x be an indeterminate, let I be an ideal of $R[x]$, and let $c(I)$ denote the ideal of R generated by the coefficients of the elements of I. The purpose of this article is to establish Proposition 4 (6) which states that under suitable conditions $R[x]/I$ is a free (projective) R-module. These results are related to a previous result of Nagata which ensures the flatness of $R[x]/I$. The motivation for our work is the case where R is a field and where I is the ideal of $R[x]$ which is generated by the minimal polynomial for an element which is algebraic over R. All rings are commutative with identity, and act on modules from the right. Terminology and notation in this paper are consistent with [**2**]. This research was supported by a grant from the National Research Council of Canada.

LEMMA 1. *Suppose that I is a principal ideal of $R[x]$ and that $c(I)=R$; then $R[x]/I$ is a flat R-module.*

PROOF. [**3**, Theorem 1].

DEFINITION 2. Let R be a ring. R is called connected if 0 and 1 are the only elements of R which are idempotent. A ring is connected if and only if its spectrum is connected as a topological space.

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LEMMA 3. *Suppose that R is connected. Let $r \in R$, and let $K=rR+(r)^*$. Suppose that r is not equal to 0, and that r is not a unit in R. Then K is a proper ideal of R.*

PROOF. By contradiction. Suppose that $K=R$. Then $1 \in K$. Thus there exist $a \in R$ and $b \in (r)^*$ such that $1=ra+b$. Thus $r=r \cdot 1=r(ra+b)=r^2a+rb=r^2a$, since $b \in (r)^*$. Therefore $ra=r^2a^2=(ra)^2$, which shows that ra is an idempotent element. By hypothesis, $ra=0$ or $ra=1$. If $ra=1$, then r is a unit which is a contradiction. Suppose $ra=0$. Then $r=r \cdot 1=r(1-ra)=rb=0$. Thus $r=0$, which is also a contradiction.

PROPOSITION 4. *Suppose that R is connected, that I is a principal ideal of $R[x]$, and that $R[x]/I$ is a finitely generated R-module. Then $R[x]/I$ is a free R-module.*

PROOF. We divide our argument into three parts.

(1) $I \neq (0)$. For suppose that $I=(0)$. Then $R[x]/I=R[x]$. By [**4**, p. 254], x satisfies a monic polynomial equation with coefficients in R. This contradicts the fact that x is an indeterminate.

(2) *The case when R is a semiprime ring.* I is a principal ideal. By (1), I is generated by a nonzero polynomial, say $f(x)=r_mx^m+\cdots+r_1x+r_0$, $r_m \neq 0$. By Lemma 3, either r_m is a unit or $K=r_mR+(r_m)^*$ is a proper ideal. We show that r_m is a unit by assuming that K is a proper ideal of R and deriving a contradiction.

Let $\varphi: R[x] \to R[x]/I$ be the canonical surjection of rings. Let $\varphi(R)=\bar{R}$ and let $\varphi(x)=\bar{x}$. Then $R[x]/I=\bar{R}[\bar{x}]$, and $\bar{R}$ is a subring of $\bar{R}[\bar{x}]$. Furthermore since $R[x]/I$ is a finite R-module, the ring $\bar{R}[\bar{x}]$ is a finite $\bar{R}$-module. By [**4**, p. 254], $\bar{x}$ satisfies a monic polynomial equation over $\bar{R}$. Thus there exist $\bar{s}_n, \bar{s}_{n-1}, \cdots, \bar{s}_1, \bar{s}_0 \in \bar{R}$, $\bar{s}_n=\bar{1}$ such that $\sum_0^n \bar{s}_j\bar{x}^j=\bar{0}$. For each j, let s_j be a preimage of $\bar{s}_j$ under the mapping $\varphi|\bar{R}$, but insist that $s_n=1$. Let $p(x)=\sum_0^n s_jx^j$. Then $\varphi(p(x))=0$. Therefore $p(x) \in I=\ker \varphi$. Thus I contains a monic polynomial. Since I is generated by $f(x)$ we have $p(x)=f(x)g(x)$ for some polynomial $g(x)=t_qx^q+t_{q-1}x^{q-1}+\cdots+t_1x+t_0$, $t_i \in R$, $i=0, 1, \cdots, q$, $t_q \neq 0$. Clearly $m+q \geqq n$. Formal multiplication of $f(x)$ and $g(x)$ yields the following 'c's as the coefficients of $x^{m+q}, x^{m+q-1}, \cdots, x^m$:

$$\begin{aligned}
c_{m+q} &= r_mt_q, \\
c_{m+q-1} &= r_mt_{q-1} + r_{m-1}t_q, \\
c_{m+q-2} &= r_mt_{q-2} + r_{m-1}t_{q-1} + r_{m-2}t_q, \\
&\ \ \vdots \\
c_m &= r_mt_0 + r_{m-1}t_1 + \cdots + r_{m-h}t_h,
\end{aligned}$$

where $h=\min[m, q]$.

The leading coefficient of $p(x)$ is 1. Since $r_m \in K$, $c_{m+q} \in K$. Therefore $c_{m+q} \neq 1$, which implies that $c_{m+q}=0$. Also $t_q \in (r_m)^*$, so $t_q \in K$. We proceed to c_{m+q-1}. Since r_m, $t_q \in K$, $c_{m+q-1} \in K$. Therefore $c_{m+q-1} \neq 1$, which implies that $c_{m+q-1}=0$. Therefore $0=r_m c_{m+q-1}=r_m^2 t_{q-1}+r_{m-1}(r_m t_q)=r_m^2 t_{q-1}$. Thus $(r_m t_{q-1})^2=0$, which implies that $r_m t_{q-1}=0$, since R is a semiprime ring. Also $t_{q-1} \in (r_m)^*$, so $t_{q-1} \in K$.

We continue this procedure arguing as follows at each equation. The new 'c' is in K because it is a linear combination of r_m and of 't's which have been shown to lie in K. Thus it differs from 1, and therefore equals 0, as the leading coefficient of $p(x)$ is 1. Multiplication of both sides of the equation by r_m allows one to conclude that the next 't' annihilates r_m, and therefore that it lies in K. Thus $t_i \in K$, $i=0, 1, \cdots, q$. But the coefficients of $f(x)g(x)$ are linear combinations of the 't's, so all the coefficients of $f(x)g(x)$ lie in K. This contradicts the fact that $p(x)$ is a monic polynomial. Thus r_m is a unit in R.

We now argue that $R[x]/I$ is a free R-module by showing that $\{\bar{1}, \bar{x}, \bar{x}^2, \cdots, \bar{x}^{m-1}\}$ is a basis for it. $R[x]/I$ is an R-module under the action $r\varphi(f(x))=\varphi(rf(x))$. $r_m^{-1}f(x) \in I$. Therefore

$$\begin{aligned}\bar{0} &= \varphi[r_m^{-1}f(x)] \\ &= \varphi[x^m + r_m^{-1}r_{m-1}x^{m-1} + \cdots + r_m^{-1}r_1 x + r_m^{-1}r_0 \cdot 1] \\ &= \bar{x}^m + r_m^{-1}r_{m-1}\bar{x}^{m-1} + \cdots + r_m^{-1}r_1\bar{x} + r_m^{-1}r_0\bar{1}.\end{aligned}$$

As is well known, this implies that $R[x]/I$ is generated as an R-module by $\{\bar{1}, \bar{x}, \bar{x}^2, \cdots, \bar{x}^{m-1}\}$. Suppose that $\sum_{i=0}^{m-1} d_i\bar{x}^i=\bar{0}$ for some $d_i \in R$, $i=0, 1, \cdots, m-1$. Let $h(x)=\sum_{i=0}^{m-1} d_i x^i$. Then $h(x) \in I$. Since $f(x)$ generates I, there exists a polynomial $j(x) \in R[x]$ such that $h(x)=f(x)j(x)$. If $j(x)\neq 0$, then the degree of $h(x)$ is greater than or equal to m, because $f(x)$ is of degree m and because its leading coefficient is a unit. But the degree of $h(x)$ is less than m. Therefore $j(x)=0$, which implies that $h(x)=0$. Thus $s_i=0$ for all i. Thus $\{\bar{1}, \bar{x}, \bar{x}^2, \cdots, \bar{x}^{m-1}\}$ is a basis for $R[x]/I$, a free R-module.

(3) *The general case*. Let $M=R[x]/I$. Then $0\to I\to R[x]\to M\to 0$ is an exact sequence of R-modules. Since M is finitely generated, I contains a monic polynomial. Therefore $c(I)=R$, which implies that M is a flat R-module by Lemma 1. Let N be the prime radical of R, and tensor the above sequence with R/N to obtain the sequence

$$(\#) \qquad 0 \to I \otimes R/N \to R[x] \otimes R/N \to M \otimes R/N \to 0.$$

By [**1**, Proposition 4, p. 30], (#) is exact. (#) is a sequence of R/N-modules. Also $I\otimes R/N \simeq I/IN$, $R[x]\otimes R/N \simeq R[x]/R[x]N \simeq (R/N)[x]$ and

$M \otimes R/N \cong M/MN$ as R/N-modules. We rewrite (#) as the exact sequence

$$0 \to I/IN \to (R/N)[x] \to M/MN \to 0.$$

R/N is a semiprime ring. Since R is connected, so is R/N by [**1**, Corollary 1, p. 132]. It is easy to verify that I/IN is principal and that M/MN is finitely generated (as R/N-modules). Therefore by the semiprime case M/MN is a finitely generated free R/N-module.

Let r be the rank of $(M/MN)_{R/N}$. By Nakayama's lemma, a basis for M/MN can be pulled back to a subset G of M of cardinality r, which generates M as an R-module. Let F be a free R-module of rank r. By mapping a basis for F onto G and extending, one obtains a surjection $F \to M$ of R-modules. Let K be its kernel. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & K & \to & F & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K/KN & \to & F/FN & \to & M/MN & \to & 0. \end{array}$$

The second row is obtained by tensoring the first by the R-module R/N, and making the usual identifications. The vertical maps are canonical.

The first row is exact by construction, and the second row is exact because M is a flat R-module. All of the squares commute.

$$F/FN \cong F \otimes R/N \cong \left(\bigoplus_1^r R\right) \otimes R/N \cong \bigoplus_1^r (R \otimes R/N) \cong \bigoplus_1^r R/N,$$

a free R/N-module of rank r. But M/MN is free of rank r over R/N. Therefore $K/KN=0$, which implies that $K \subset FN$ by commutativity. By [**1**, 23(d), p. 66], $K=0$. Therefore $M \cong F$, a free module.

Definition 5. Let R be a ring. R is a semiconnected ring if $|B(R)|$ is finite. This terminology is chosen in analogy with the definition of a semilocal ring as a ring in which the number of maximal ideals is finite (a ring is local if the number of maximal ideals is minimal).

A straightforward argument shows that a semiconnected ring is isomorphic to a finite direct product of connected rings, and that a module over a semiconnected ring is canonically a finite product of submodules over these connected rings. As a consequence Proposition 4 can be generalized as follows:

Proposition 6. *Let R be a semiconnected ring, let I be a principal ideal of $R[x]$ and let $M=R[x]/I$ be a finite R-module. Then M_R is projective.*

Let R be a ring and let S be an overring of R. Recall that an element $a \in S$ is integral over R if a satisfies a monic polynomial equation with coefficients from R.

PROPOSITION 7. *Let R be a ring and let $R[a]$ be an integral extension of R. Let x be an indeterminate and let I be the kernel of the canonical surjection $R[x]\to R[a]$. Assume that I is a principal ideal of $R[x]$. Then $R[a]_R$ is flat. Furthermore $R[a]_R$ is free if R is connected, and is projective if R is semi-connected.*

PROOF. If $a=0$, then $R[a]=R$, which is a free R-module. Suppose that $a\neq 0$. Since $R[a]$ is an integral extension of R, there exist $r_i\in R$, $i=0, 1,\cdots,n$, $r_n=1$, such that $\sum_{i=0}^n r_i a^i=0$. By [**4**, p. 254], $(R[a])_R$ is finite. Also since $\sum_0^n r_i x^i\in I$ and $r_n=1$, $c(I)=R$. The result now follows from Lemma 1 and Propositions 4 and 6.

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