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COMPLETELY CYCLIC INJECTIVE SEMILATTICES

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ABSTRACT. We characterize semilattices S with identity for which every cyclic S-system is injective. We note that this condition, unlike the R-module case, is not equivalent to the condition that every S-system is injective.

1. **Introduction.** A (right) S-system is a set M acted on by a semigroup S such that $m(s_1s_2)=(ms_1)s_2$ for all $m \in M$ and all $s_1, s_2 \in S$. We shall call a semigroup S completely injective if every S-system is injective and completely cyclic injective if every cyclic S-system is injective. In **[4]** B. Osofsky showed that these two notions coincide when they are formulated for R-modules. In the present paper we characterize completely cyclic injective semilattices and then use a characterization of completely injective semilattices due to Feller and Gantos **[2]** to show that these notions fail to coincide for S-systems. Finally, we discuss a generalization of our characterization suggested by the principal result of **[2]**.

2. **Completely cyclic injective semilattices.** If R and M are S-systems, a map $\phi: R \to M$ is an *S-homomorphism* when $\phi(r)s=\phi(rs)$ for all $r \in R$ and all $s \in S$. An S-system M is *injective* when every S-homomorphism from an S-subsystem P of an S-system R into M has an extension to all of R. A semigroup S with identity is *completely injective* when every unitary S-system is injective. An S-system M is *cyclic* when there exists an $m \in M$ such that $M=mS$. Finally, we define a semigroup S to be *completely cyclic injective* when every cyclic S-system is injective.

Throughout the remainder of this section S will denote a semilattice with identity element 1. If $\sim$ is a semilattice congruence on S it is clear that $S/\sim$ is an S-system with $[x]s=[xs]$ and that it is cyclic. Thus it can be shown that every semilattice homomorphic image of S is a cyclic S-system.

LEMMA. *Every cyclic S-system is isomorphic (as an S-system) to a semilattice homomorphic image of S.*

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PROOF. Let mS be a cyclic S-system. Define a relation $\sim$ on S by $x \sim y$ if and only if $mx = my$. It is easy to see that $\sim$ is a semilattice congruence on S and that the map $mx \mapsto [x]$ is an S-isomorphism between mS and $S/\sim$.

LEMMA. *A semilattice homomorphic image T of S is injective (in the category of S-systems) if and only if it is a complete lattice satisfying the following distributive law*

$$(1) \qquad s \wedge \bigvee A = \bigvee (s \wedge a \mid a \in A) \quad \textit{for all } s \in T \textit{ and all } A \subseteq T.$$

PROOF. This follows from Corollary 4 of [3].

DEFINITION. A *chain of diamonds* is a lattice in which a, b, c mutually distinct, $a \nleqq b$, and $b \nleqq a$ imply that either $c < a$ and $c < b$, or $a < c$ and $b < c$. (We shall write $a|b$ when $a \nleqq b$ and $b \nleqq a$.)

If I is an ideal of the semilattice S we recall that the Rees factor semigroup S/I is a semilattice homomorphic image of S in which the elements of I are identified as the zero of S/I. This is used repeatedly in the following proof along with the fact that every semilattice homomorphic image of S is a cyclic S-system and hence a distributive lattice whenever it is injective.

THEOREM. *S is completely cyclic injective if and only if it is a complete (as a lattice) chain of diamonds.*

PROOF. Suppose that every cyclic S-system is injective. S itself is then injective and hence a complete lattice. We must show that it is a chain of diamonds. We will use N_5 to denote the five element nonmodular lattice and M_5 to denote the remaining five element nondistributive lattice.

Let us suppose that $a, b, c \in S$ are mutually distinct and that $a|b$. We first assume that $c|a$ and $c|b$ and we consider the three cases determined by the cardinality of $A = \{a \vee b, a \vee c, b \vee c\}$. If $|A| = 1$, we identify the elements of the ideal $[0, a) \cup [0, b) \cup [0, c)$ to get a homomorphic image of S which has as a sublattice $\{0, a, b, c, a \vee b \vee c\} \cong M_5$, a contradiction of distributivity. If $|A| = 2$, we may suppose that $b \vee c < a \vee b = a \vee c$ and identify the elements of $[0, a) \cup [0, b) \cup [0, c)$ to get a homomorphic image which has as a sublattice $\{0, a, b, b \vee c, a \vee b\} \cong N_5$, again a contradiction of distributivity. If $|A| = 3$, we consider subcases determined by the number of comparabilities in A. If there are none (i.e., if the elements of A are mutually noncomparable) we identify the elements of $[0, a \vee b) \cup [0, a \vee c) \cup [0, b \vee c)$ to get a homomorphic image having as a sublattice $\{0, a \vee b, a \vee c, b \vee c, a \vee b \vee c\} \cong M_5$, a contradiction. Exactly one comparability in A is not possible since, for example, $a \vee b \leqq a \vee c$ implies that $b \leqq a \vee c$, which implies that $b \vee c \leqq a \vee c$. The only way that exactly two comparabilities can exist without transitivity or the argument of the preceding sentence producing a third is to have one element of A larger than the other two, say $a \vee b$, $b \vee c < a \vee c$. In this case we get a contradiction by identifying the elements of $[0, a) \cup [0, c)$ to get a homomorphic image having as a sublattice $\{0, a, c, b \vee c, a \vee c\} \cong N_5$. Finally, three comparabilities in A are not possible since, for example, $a \vee b \leqq b \vee c \leqq a \vee c$ implies that $a \vee c \leqq b \vee c$, which implies that $a \vee c = b \vee c$. Thus we have eliminated the possibility that $c|a$ and $c|b$.

If $c < b$ we have either $a = a \vee c$ or $a \vee c = a \vee b$. This is true since we have $a \leqq a \vee c \leqq a \vee b$, and if $a < a \vee c < a \vee b$ we can identify the elements of $[0, a) \cup [0, b)$ to get a homomorphic image having as a sublattice $\{0, b, a, a \vee c, a \vee b\} \cong N_5$. We can eliminate the possibility that $a \vee c = a \vee b$ by considering that in such a case the elements of $[0, a) \cup [0, c)$ could be identified to give a homomorphic image having as a sublattice $\{0, a, c, b, a \vee b\} \cong N_5$. Thus if $c < b$ we must have $a = a \vee c$ and hence $c < a$. Finally, suppose that $b < c$. By the preceding argument, $a|c$ would give $b < a$, a contradiction, so either $a < c$ or $c < a$. Since $c < a$ would give $b < a$ by transititivity, we must have $a < c$. This completes the proof that S is a chain of diamonds.

To prove the converse, let us suppose that S is a complete chain of diamonds. We must show that every semilattice homomorphic image of S is a complete lattice satisfying (1). Let $f: S \to T$ be a semilattice epimorphism. Since f is isotone it is clear that if T is a lattice it must be a chain of diamonds. To show that T is a complete lattice it will suffice to show that arbitrary joins exist in T. Let $\{f(s_\alpha)\}$ be a subset of T. If $\{f(s_\alpha)\}$ has a maximum element then it clearly has a join so let us suppose that it fails to have a maximum element. Let $s = \bigvee s_\alpha$. Clearly $f(s_\alpha) \leqq f(s)$ for all α. To show that $f(s)$ is the desired join we suppose that $f(s_\alpha) \leqq f(u)$ for all α. We wish to show that $f(x) \leqq f(u)$. There cannot exist an s_β such that $u \leqq s_\beta$, for then we would have $f(u) \leqq f(s_\beta) \leqq f(u)$ and $\{f(s_\alpha)\}$ would have a maximum element. Suppose there exists an s_β such that $u|s_\beta$. Then if $\alpha \neq \beta$, either $s_\alpha \leqq u \wedge s_\beta$ or $u \vee s_\beta \leqq s_\alpha$ since S is a chain of diamonds. The second of these would give $u \leqq s_\alpha$, a possibility we have eliminated. Thus $s_\alpha \leqq u \wedge s_\beta$ for all $\alpha \neq \beta$ and we have $f(s_\alpha) \leqq f(u \wedge s_\beta) = f(u) \wedge f(s_\beta) \leqq f(s_\beta)$ for all $\alpha \neq \beta$, giving $\{f(s_\alpha)\}$ a maximum element again. The only remaining possibility is that $s_\alpha \leqq u$ for all α and thus $s = \bigvee s_\alpha \leqq u$, giving $f(s) \leqq f(u)$. Thus T is a complete lattice and it only remains to show that a complete chain of diamonds satisfies (1). We must show that $a \wedge \bigvee b_\alpha \leqq \bigvee (a \wedge b_\alpha)$ and to do this we consider three cases. First, if $a \leqq b_\beta$ for some β then $a \wedge \bigvee b_\alpha = a \leqq \bigvee (a \wedge b_\alpha)$. Second, if $a|b_\beta$ for some b_β then for each $\alpha \neq \beta$ either $b_\alpha < b_\beta$ or $b_\beta < b_\alpha$. If $b_\beta < b_\alpha$ for some $\alpha \neq \beta$, then $a|b_\beta$ gives $a < b_\alpha$ and we are back in the first case. If $b_\alpha < b_\beta$ for all $\alpha \neq \beta$, then $\bigvee b_\alpha = b_\beta$ and we have $a \wedge \bigvee b_\alpha = a \wedge b_\beta \leqq \bigvee (a \wedge b_\alpha)$. Finally, if $b_\alpha \leqq a$ for all α, then $\bigvee b_\alpha \leqq a$ and we have $a \wedge \bigvee b_\alpha = \bigvee b_\alpha = \bigvee (a \wedge b_\alpha)$. Thus a complete chain of diamonds satisfies (1) and we have shown that every cyclic S-system is injective.

Theorem 2.9 of [**2**] says that a semigroup with central idempotents is completely injective if and only if it is a semilattice of groups whose idempotents form a complete dually well-ordered chain. Thus completely injective semilattices may be characterized as complete dually well-ordered chains and it is clear, in light of our theorem, that a completely cyclic injective semilattice need not be completely injective.

3. **Semigroups with central idempotents.** In light of Feller and Gantos' result one might conjecture that the completely cyclic injective semigroups with central idempotents are the semilattices of groups whose idempotents form a complete chain of diamonds. We show that this is not the case by offering a counterexample.

First we recall that an S-system M is *weakly injective* if for any right ideal I of S and S-homomorphism $\phi: I \to M$ there exists an $m \in M$ such that $\phi(x) = mx$ for all $x \in I$. It is shown in [**1**, p. 264] that an injective S-system is weakly injective.

Let $G = \{e, x\}$ be the cyclic group of order two with identity e and let $S = G \cup \{f\} \cup \{0\} \cup \{1\}$ where 1 and 0 are the identity and zero of S and $f^2 = f$. Define $fG = 0 = Gf$. Then S is a semilattice of groups whose idempotents are a diamond. Consider the ideal $I = G \cup \{f\} \cup \{0\}$ and define $\phi: I \to S$ by $\phi(x) = e$, $\phi(e) = x$, $\phi(f) = f$, and $\phi(0) = 0$. Then ϕ is an S-homomorphism but is not realized by any element of S. Hence S is not completely cyclic injective since S itself is not even weakly injective.

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