PERIODIC SOLUTIONS OF SMALL PERIOD OF SYSTEMS
OF nTH ORDER DIFFERENTIAL EQUATIONS

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Abstract. This paper consists of a study of the existence of periodic solutions of systems of nth order ordinary differential equations using tools from degree theory.

1. Introduction. Let $I$ denote the compact interval $[0, T]$, $R = (-\infty, \infty)$ and let $f : I \times R^{nm} \to R^m$ be continuous. We consider the differential system

\[(1.1) \quad x^{(n)} + f(t, x, x', \ldots, x^{(n-1)}) = 0,\]

and give conditions which ensure that (1.1) has periodic solutions of small period, i.e., conditions which ensure the existence of a constant $\omega_0$, $0 < \omega_0 \leq T$, such that for every $\omega$, $0 < \omega \leq \omega_0$, (1.1) has a solution $x(t)$ such that

\[(1.2) \quad x^{(i)}(0) = x^{(i)}(\omega), \quad i = 0, \ldots, n-1.\]

The considerations in this paper are largely motivated by a paper of Seifert [7] where degree-theoretic arguments are used to prove the existence of periodic solutions (of small period) for the undamped oscillator

\[(1.3) \quad x'' + g(x) = p(t),\]

where $g(x)$ is a general restoring force having the property that

\[(1.4) \quad xg(x) > 0, \quad x \neq 0, \quad |g(x)| \to \infty \quad \text{as} \quad |x| \to \infty.\]

Using only simple results from ordinary differential equations and from degree theory (we use the theory as developed in [2] and [6]) we establish the following general principle.

Theorem 1. Let there exist a nonempty bounded open set $A \subset R^m$ such that $f(0, x, 0, \ldots, 0) \neq 0$ for $x \in \partial A$ and let $\deg(f(0, x, 0, \ldots, 0), A, 0) \neq 0$. Then there exists $\omega_0$, $0 < \omega_0 \leq T$, such that for every $\omega$, $0 < \omega \leq \omega_0$, (1.1) has a solution satisfying the periodic boundary conditions (1.2).

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Throughout the paper, we shall say that a differential equation has periodic solutions of small periods whenever the conclusion of Theorem 1 holds for that equation. Furthermore, we assume throughout that all solutions of the equation being considered exist on the basic interval \([0, T]\), and \(\|\cdot\|\) shall denote the Euclidean norm in \(\mathbb{R}^m\). No ambiguity will arise if we use the symbol 0 for the zero of all Euclidean spaces considered in this paper.

2. Corollaries.

**Corollary 1.** Let \(f : I \times \mathbb{R}^{nm} \rightarrow \mathbb{R}^m\) be continuous and let there exist a constant \(r > 0\) such that either

\(\text{(i) } x \cdot f(0, x, 0, \ldots, 0) > 0, \|x\| = r, \text{ or}\)

\(\text{(ii) } x \cdot f(0, x, 0, \ldots, 0) < 0, \|x\| = r.\)

Then (1.1) has periodic solutions of small periods.

**Proof.** Let \(A = \{x \in \mathbb{R}^m : \|x\| < r\}\). Then either condition (i) or (ii) above implies that \(\text{deg}(f(0, x, 0, \cdots, 0), A, 0)\) is defined. Further, either of the conditions implies that the vector field \(f(0, x, 0, \cdots, 0), x \in \partial A\), has the property that \(f(0, x, 0, \cdots, 0)\) and \(f(0, -x, 0, \cdots, 0)\) do not have the same direction for every \(x \in \partial A\) and hence that \(f(0, x, 0, \cdots, 0)\) is homotopic to an odd vector field. It follows from the homotopy invariance theorem of degree theory (see [2] or [6]) and from Borsuk’s theorem (see [6]) that \(\text{deg}(f(0, x, 0, \cdots, 0), A, 0) \neq 0\). We may, therefore, apply Theorem 1.

**Corollary 2.** Consider the differential equation

\(x^{(n)} + h(t, x, x', \cdots, x^{(n-1)}) + g(x) = p(t),\)

where \(h : I \times \mathbb{R}^n \rightarrow \mathbb{R}, g : \mathbb{R} \rightarrow \mathbb{R}, p : I \rightarrow \mathbb{R}\) are continuous and have the property that \(h(0, x, 0, \cdots, 0) = 0, xg(x) > 0 (< 0)\) and \(|g(x)| \rightarrow \infty\) as \(|x| \rightarrow \infty\). Then (2.1) has periodic solutions of small periods.

**Proof.** Since \(p\) is continuous, \(xg(x) > 0 (< 0)\) and \(|g(x)| \rightarrow \infty\) as \(|x| \rightarrow \infty\), there will exist a constant \(r > 0\) such that either

\(\text{(i) } x(g(x) - p(0)) > 0, |x| = r, \text{ or}\)

\(\text{(ii) } x(g(x) - p(0)) < 0, |x| = r.\)

We may, therefore, apply the previous corollary to complete the proof.

**Remark.** Taking \(h \equiv 0, n = 2\), we obtain the result of Seifert [7]. Also, if \(n = 2\) and \(h = k(x, x')\) we obtain the existence of periodic solutions of small period for the forced Lienard equation \(x'' + k(x, x') + g(x) = p(t)\).

**Remark.** In the case of equation (2.1), it will be apparent from the proof of Theorem 1 that the constant \(\omega_0\) depends only on the left side of the equation and a bound on \(|p(t)|, 0 \leq t \leq T\). Thus if \(p(t)\) is a periodic function (in the usual sense) of period \(\omega, \omega \leq \omega_0\), then the corresponding
periodic solution (whose existence is guaranteed by Theorem 1) may be extended periodically so as to yield a periodic solution (in the usual sense) of (2.1). A similar remark also holds for equation (1.1).

**Corollary 3.** Consider the nth order scalar equation

\[ x^{(n)} + f(t, x, x', \ldots, x^{(n-1)}) = 0, \]

where \( f: I \times \mathbb{R}^n \to \mathbb{R} \) is continuous, and let there exist constants \( \alpha, \beta, \alpha < \beta \), such that either

(i) \( f(t, \alpha, 0, \ldots, 0) < 0 < f(t, \beta, 0, \ldots, 0) \), or

(ii) \( f(t, \alpha, 0, \ldots, 0) > 0 > f(t, \beta, 0, \ldots, 0) \),

with strict inequalities holding in either case for \( t=0 \). Then (2.2) has periodic solutions of small periods.

**Proof.** In either case, \( \deg(f(0, x, 0, \ldots, 0), (\alpha, \beta), 0) \neq 0 \).

**Remark.** Corollary 3 represents generalizations of some results in [4] and [5]. Case (i) extends Theorem 1 of [5] in the sense that no local Lipschitz condition is required and further that \( n \) need not equal 2. On the other hand, we need to assume here that strict inequalities hold for \( t=0 \) in order for \( \deg(f(0, x, 0, \ldots, 0), (\alpha, \beta), 0) \) to be defined. Case (ii) extends some special cases of results in [4] to higher order equations; however, the results in [4] are valid for arbitrary periods whereas Corollary 3 only guarantees the existence of solutions of small periods.

3. **Proof of Theorem 1.** Before proceeding with the proof, we need some terminology and some preliminary lemmas.

Let \( y = (x, x', \ldots, x^{(n-1)}) \), \( F(t, y) = (-x', \ldots, -x^{(n-1)}, f(t, y)) \), \( k = nm \), and consider the equivalent system of differential equations

\[ y' + F(t, y) = 0. \]

A point \( y_0 \in \mathbb{R}^k \) is called an \( \omega_0 \)-nonrecurrence point of (3.1) if every solution \( y(t) \) of (3.1) with \( y(0) = y_0 \) is such that \( y(t) \neq y_0 \) for \( 0 < t \leq \omega_0 \).

**Lemma 1.** Let \( \Omega \subseteq \mathbb{R}^k \) be a nonempty bounded open region whose boundary \( \partial \Omega \) consists of \( \omega_0 \)-nonrecurrence points only, \( \omega_0 \leq T \). Further, let \( \deg(F(0, y), \Omega, 0) \) be defined and nonzero. Then for every \( \omega, 0 < \omega \leq \omega_0 \), there exists a solution \( y(t) \) of (3.1) such that \( y(0) = y(\omega) \).

**Proof.** This follows from the results of Krasnosel'skii [3, pp. 79–83] and the observation that \( \deg(F(0, y), \Omega, 0) \neq 0 \) implies the nonvanishing of the rotation of the vector field considered by Krasnosel'skii.

In the sequence of lemmas to follow, we shall show that the hypotheses of Theorem 1 allow us to construct a region \( \Omega \) and find a number \( \omega_0 \) so that Lemma 1 may be applied to equation (3.1).
**Lemma 2.** Let $N > 0$ be given and let $B \subseteq \mathbb{R}^m$ be a bounded open set such that $A \subseteq B$. Then there exists $0 < \omega_1 \leq T$, such that every solution $x(t)$ of (1.1) with $x^{(i)}(0) = r_i$, $i = 0, \ldots, n-1$, $r_i \in A$, $\|r_i\| \leq N$, $i = 1, \ldots, n-1$, has the property that $x(t) \in B$, $\|x^{(i)}(t)\| \leq 2N$, $i = 1, \ldots, n-1$, $0 \leq t \leq \omega_1$.

**Proof.** This is an immediate consequence of the continuity of $f$ and the basic initial value problem existence results.

For $N$ and $B$ as defined above we let

$$\Omega = \{(x, x', \cdots, x^{(n-1)}): x \in A, \|x^{(i)}\| < N, i = 1, \ldots, n-1\},$$

(3.2)

$$\Omega = \{(x, x', \cdots, x^{(n-1)}): x \in B, \|x^{(i)}\| < 2N, i = 1, \ldots, n-1\},$$

and for every $K \subseteq \Omega$ we let $S(K)$ denote the set of all solutions $y$ of (3.1) with $y(0) \in K$. Lemma 2 thus implies that if $y \in S(K)$ then $y(t) \in \Omega$ for $0 \leq t \leq \omega_1$.

**Lemma 3.** For every $K \subseteq \Omega$, $S(K)$ has compact closure in $C([0, \omega_1], \mathbb{R}^k)$.

**Proof.** An application of the Ascoli-Arzela theorem.

**Lemma 4.** There exists $\omega_0$, $0 < \omega_0 \leq \omega_1$, such that $\partial \Omega$ consists only of $\omega_0$-nonrecurrence points of (3.1).

**Proof.** Assume the contrary. Then there is a sequence $(t_n)_{n=1}^\infty$, $t_n \to 0$ as $n \to \infty$, and a sequence of points $y_n \in \partial \Omega$ such that (3.1) has a solution $y_n(t)$ with $y_n(0) = y_n(t_n)\). By Lemma 2, $y_n(t) \in \Omega$, $0 \leq t \leq \omega_1$. By passing to subsequences, if necessary, relabeling, and applying Lemma 3, we may assume that $\lim_{n \to \infty} y_n = y^* \in \partial \Omega$ and that $\lim_{n \to \infty} y_n(t) = y(t)$ is a solution of (3.1) with $y(0) = y^*$. Since $y_n(t) = y_n - \int_0^t F(s, y_n(s)) \, ds$ we conclude that

$$\frac{1}{t_n} \int_0^{t_n} F(s, y_n(s)) \, ds = 0.$$ 

By continuity of $F$ we have

$$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \|F(s, y_n(s)) - F(0, y^*)\| \, ds = 0,$$

and therefore that $F(0, y^*) = 0$.

Let $y^* = (x, x', \cdots, x^{(n-1)})$, then

$$F(0, y^*) = 0 = (-x', \cdots, -x^{(n-1)}, f(0, y^*),$$

which implies that $f(0, x, 0, \cdots, 0) = 0$ for some $x \in \partial A$, a contradiction.
Proof of Theorem 1. We now apply Lemma 1 with \( \omega_0 \) as chosen in Lemma 4 and \( \Omega \) as given by (3.2). We only need to verify that 
\[ \text{deg}(F(0, y), \Omega, 0) \neq 0. \]

Since 
\[ \text{deg}(F(0, y), \Omega, 0) = \text{deg}(-x', \cdots, -x^{(n-1)}, f(0, x, x', \cdots, x^{(n-1)}), \Omega, 0), \]

and since \( f(0, x, 0, \cdots, 0) \neq 0 \) on \( \partial A \), we conclude that \( \text{deg}(F(0, y), \Omega, 0) \) is defined. Further \((-x', -x^n, \cdots, -x^{(n-1)}, f(0, x, \lambda x', \cdots, \lambda x^{(n-1)})) \neq 0 \) on \( \partial \Omega, 0 \leq \lambda \leq 1 \); hence \( F(0, y) \) is homotopic to \((-x', \cdots, -x^{(n-1)}, f(0, x, 0, \cdots, 0)). \) By the homotopy invariance theorem of degree theory (see [2] or [6])
\[ \text{deg}(F(0, y), \Omega, 0) = \text{deg}(-x', \cdots, -x^{(n-1)}, f(0, x, 0, \cdots, 0), \Omega, 0). \]
The latter, on the other hand, is nonzero, if and only if 
\[ \text{deg}(f(0, x, 0, \cdots, 0), A, 0) \neq 0 \]
(see [6]). This completes the proof.

Remark. We note from the above lemmas and proofs that \( \omega_0 \) depends on the arbitrarily chosen constant \( N \) and region \( B \). Thus in varying both of these quantities one may possibly increase \( \omega_0 \) and hence increase the range of possible periods for solutions of (1.1) satisfying the periodic boundary conditions (1.2). The interested reader is referred to the paper [1] where existence results for periodic solutions (of period \( T \)) of systems of second order equations are established using methods similar to the ones used in this paper.

References