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KRONECKER SETS AND METRIC PROPERTIES OF M_0-SETS

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ABSTRACT. A method for constructing both sets of multiplicity and Kronecker sets within a given set of multiplicity is derived from the work of Ivashev-Musatov; it is shown that the Hausdorff measures and other measures are essentially distinct. Finally, an improvement of a theorem of Salem is obtained, using Pyateckiĭ-Shapiro's theorem on non-M sets.

1. The class of complex Borel measures μ on the real axis, such that $\hat{\mu}(u)\to 0$ as $|u|\to\infty$, is denoted R, and a closed set is called M_0 if it supports some measure $\mu\neq 0$ of class R. With a measure μ the entire space $L^1(\mu)$ is contained in R [7, I, p. 143], so that we mostly study probability measures in R. The most striking examples of non-M_0 sets are the Kronecker sets: A compact linear set E is a K-set if each continuous function on E of modulus 1 admits uniform approximation on E by characters $\chi(x)=e^{iux}$ $(-\infty<u<\infty)$ ([**2**], [**6**]).

An easy application of Fubini's theorem shows that if E_1 and E_2 are closed, E_1 is M_0 and $m(E_2)>0$, then $E_1\cap(E_2+x)$ is M_0 for an x-set of positive Lebesgue measure. A sharp converse is true.

THEOREM 1. *Let F be a closed set of Lebesgue measure $m(F)=0$. Then there exists an M_0-set E so that $(F+x)\cap E$ is a K-set for every real x.*

Theorem 1 is a consequence of a more general assertion; for closed sets F and numbers $r>0$ let $m(F,r)=m\{x:\operatorname{dist}(x,F)\leqq r\}$.

THEOREM 1′. *Let g be any positive, increasing function on* $(0,1)$ *and let $g(0+)=0$. Then there is an M_0-set E such that each closed set $F\subseteq E$ is a K-set, unless $m(F,r)>g(r)$ for all $r\leqq r_0(F)$.*

Theorem 1′ is derived from a very general theorem in which Lebesgue measure plays no special role.

THEOREM 2. *Let μ be a probability of class R and of closed support S; let $(T_k)_1^\infty$ be a sequence of closed sets with* $\lim \mu(T_k)=0$. *Then there exists an*

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M_0-set $S_0 \subseteq S$ with this property: for every infinite set I of integers k, $S_0 \cap \bigcap_I T_k$ is a K-set.

The proof of Theorem 2 requires a few preliminary remarks and adjustments.

(a) There are open sets $W_k \supseteq T_k$ whose boundaries ∂W_k have μ-measure 0, and moreover $\mu(W_k) < \mu(T_k) + k^{-1}$. This much is true for Baire measures in any metric space. In the set of probability measures in $L^1(\mu)$, the mappings $\lambda \to \lambda(W_k)$ are w^*-continuous, because $\mu(\partial W_k) = 0$. In the same direction, let V be an open set; the set of probabilities such that $\lambda(V) = 0$ is w^*-closed.

(b) A distance function on the Borel measures is defined by the formula $p(\lambda_1, \lambda_2) = \sup |\lambda_1(W_k) - \lambda_2(W_k)|$. In the proof we shall construct probability measures μ_j in $L^1(\mu)$ such that $\mu = \mu_0$ and

(i) $p(\mu_j, \mu_{j+1}) \leqq 2^{-j}$ for $j = 0, 1, 2, \cdots$.

(ii) $|\hat{\mu}_j(u) - \hat{\mu}_{j+1}(u)| \leqq 2^{-j}$ for every u, with at most one exception $j = j(u)$.

(iii) $\|\mu_{j+1} - \mu_j\| \leqq 2\mu_j(W_j)$.

From (i) it will follow that $\lim \mu_j(W_j) = 0$ and then from (ii) and (iii) that $\hat{\mu}_j$ converges uniformly on the real axis, to a function also tending to 0 at infinity. The property $\mu(\partial W_k) = 0$ is used for (i).

(c) Let λ be a probability measure of class R, and f a real measurable function. Then the measurable functions of x, $ux - f(x)$, tend to uniform distribution (modulo 1) in λ-measure, as $|u| \to \infty$. To prove this we use Weyl's criterion [**1**, IV] for uniform distribution, with the notation $e(t) \equiv \exp 2\pi i t$. It must be proved that

$$\lim \int e(kux - kf(x))\, \lambda(dx) = 0 \quad \text{for } k = 0, 1, 2, \cdots.$$

But the complex measures $e(-kf(x))\, \lambda(dx)$ are of class R and this fact expresses the limit relation required.

(d) Let now S be compact and totally disconnected, as is easily achieved, and $(g_k)_1^\infty$ be a dense sequence in the real Banach space $C(S)$. We shall use the fact that, on S and any closed subset of S, a continuous function of modulus 1 admits uniform approximation by the subset $e(g_k)$. We construct closed sets $A_j \subseteq W_j$ and measures μ_j by the formulas

$$\mu_{j+1}(B \sim W_j) \equiv \mu_j(B \sim W_j), \qquad \mu_{j+1}(B \cap W_j) \equiv c_j \mu_j(B \cap A_j)$$

for appropriate constants $c_j > 0$, so that μ_{j+1} is again a probability. As μ_{j+1} and μ_j differ only by a certain adjustment outside of W_j, an easy calculation yields (iii). Moreover, to each function $g_1, \cdots, g_j$ there will be integers $u_1, \cdots, u_j$ so that $|u_1(x) - g_1(x)| < 2^{-j}$ (modulo 1), $|u_j(x) - g_j(x)| < 2^{-j}$ (modulo 1), on A_j. Observe also that the measures μ_j have decreasing closed supports; this, with the last remark in (a), shows that any w^*-accumulation point λ of the sequence (μ_j) has support S_0 such that $S_0 \cap T_j \subseteq S_0 \cap W_j \subseteq A_j$ so that S_0 has the 'Kronecker' properties asserted.

The measures μ_j are constructed by induction; suppose this accomplished up to μ_s. Because $\mu_0, \cdots, \mu_s$ are of class R (belonging to $L^1(\mu)$), there is a number Y such that $|\hat{\mu}_j(u)| < 2^{-s-1}$ for $0 \leqq j \leqq s$ whenever $|u| \geqq Y$. Thus requirement (ii) need only be attained on the interval $|u| \leqq Y$.

(e) Let $0 < \delta < 2^{-s} Y^{-1}$, and let (W_n) be a finite partition of W_s into subsets of diameter $< \delta$. When the measure μ_{s+1} has the property that $\mu_{s+1}(W_n) = \mu_s(W_n)$ for each n, then $|\hat{\mu}_{s+1}(u) - \hat{\mu}_s(u)| \leqq \delta |u| < 2^{-s}$ on $[-Y, Y]$. It remains to construct the set A_s and attain inequality (i).

Using Weyl's criterion s times in succession, we find that there are integers $u_1, \cdots, u_s$ so that the set $\{|u_j x - g_j(x)| < 2^{-s}$ (modulo 1), $1 \leqq j \leqq s\}$ meets each W_n in a set of μ_s-measure at least $2^{-s-1}\mu_s(W_n) = a_s \mu_s(W_n)$. Here a_s depends only on s. Then, using the regularity of Borel measures, there is a closed set A_s on which the required approximations are valid, while $\mu_s(W_n \cap A_s) = \frac{1}{2} a_s \mu_s(W_n)$. Thus μ_{s+1} can be defined with $c_s = 2a_s^{-1}$ (or, in fact, any number $c_s > 2^s$).

(f) To obtain inequality (i) we note that all the measures μ_{s+1} arising in this construction satisfy the same inequality $0 \leqq \mu_{s+1} \leqq c_s \mu_s$ (as set functions), whence $\mu_{s+1}(W_k) \leqq c_s \mu_s(W_k)$ for all measures μ_{s+1}. Thus the inequality $|\mu_{s+1}(W_k) - \mu_s(W_k)| \leqq 2^{-s}$ is a constraint only when $c_s \mu_s(W_k) > 2^{-s-1}$, that is for $k \leqq k_s$, say. For these values of k, we use the w^*-continuity of the mappings $\lambda \to \lambda(W_k)$ on $L^1(\mu)$, by making $\delta > 0$ small enough to bring each μ_{s+1} close to μ_s in the w^*-topology. This completes the construction of μ_{s+1} and the proof of Theorem 2. It is worthwhile remarking that if any sequence $e(u_m x)$ of characters is assigned in advance, with $|u_m| \to \infty$, then the Kronecker property of the intersections $S_0 \cap \bigcap_I T_k$ can be enforced using only these characters.

PROOF OF THEOREM 1′. Let S be an interval of length 1 and μ the Lebesgue measure on S. Let $J(n)$ be the usual division of S into intervals of length n^{-1}, $C(n)$ the set of unions of at most $ng(2n^{-1})$ of these intervals, and finally let $(T_k)_1^\infty$ be an enumeration of $C(n)$. To see that Theorem 2 applied with the present choice of μ and (T_k) yields Theorem 1′, consider a subset $F \subseteq S$ such that $m(F, r) \leqq g(r)$ for some r in $(0, 1)$. Defining n by the inequalities $n^{-1} < r \leqq (n-1)^{-1}$, we see that F meets at most $nm(F, r) \leqq ng(r) \leqq ng(2n^{-1})$ intervals from $J(n)$, and Theorem 1′ is proved.

To see the special properties of the measure function $m(F, r)$, we compare it with Hausdorff measure [5, II]. Recall that if h is increasing and positive on $(0, \infty)$ and $h(0+) = 0$, a set F has *h-measure* 0 if, to each $\varepsilon > 0$, there is a division $F = UF_i$, so that $\sum h(\text{diam } F_i) < \varepsilon$. By almost the same arguments as before, the following improvement of the theorem of Ivashev-Musatov is obtained [3].

THEOREM 3. *Each M_0-set E contains an M_0-set E_1 of h-measure* 0.

2. Theorems 1′ and 3 are in obvious contrast, and another contrast to Theorem 3 is obtained from Dirichlet's theorem on simultaneous approximation ([**1**, I]; [7, II, p. 153]).

THEOREM 4. *Let μ be a probability of class R and F_k any union of $C \log k$ intervals of length k^{-1}. Then* $\lim \mu(F_k)=0$.

PROOF. We shall first show that $\limsup \mu(F_k) \leqq 2 \exp -C^{-1}$. In fact let $\exp -C^{-1}<b<1$ and $\delta>0$, and let $x_1, \cdots, x_N$ ($N \leqq C \log k$) be the centers of the given intervals. The inequalities

$$|\xi x_j| \leqq b \text{ (modulo 1)}, \qquad 1 \leqq j \leqq N, \; |\xi| \leqq \delta k,$$

have an integer solution $\xi>0$ as soon as $N \log b+\log \delta+\log k>0$, and in fact the *largest* solution, say ξ_k, tends to $+\infty$ because $C \log b>-1$. (See the remarks below on Dirichlet's theorem.) Because $1 \leqq \xi_k \leqq \delta k$ we find $|\xi_k x| \leqq b+\delta$ (modulo 1) for every x in F_k and by Weyl's criterion, $\limsup \mu(F_k) \leqq 2(b+\delta)$.

Next, for any integer $r \geqq 1$, the intervals comprised in F_k can be divided into r approximately equal classes; this yields the estimate

$$\limsup \mu(F_k) \leqq 2r \exp -rC^{-1},$$

whence $\lim \mu(F_k)=0$.

This theorem is to be compared with Theorem 3, for it gives a sort of modulus of continuity. In particular, F is non-M_0 provided $\liminf m(F, r)/r \log r^{-1}<\infty$. This result is very sharp, as shown by a theorem of Kahane [**4**]. (Theorem 4 resolves the question raised at the conclusion of [**4**].)

In both Theorem 4 and the following argument, it is necessary to obtain a large number of solutions in Dirichlet's theorem on simultaneous approximation. The references cited deduce Dirichlet's theorem from Minkowski's theorem on lattice points in a symmetric convex set S in the Euclidean space R^p. However, if $2^{-p}V(S)$ is large, the analysis of Minkowski's theorem yields a large number of lattice points, and these determine distinct solutions in the Dirichlet theorem.

Besides the concept of M_0-set, there is also that of *M-set*, defined as follows (for closed sets, at any rate). A closed set E is an M-set if it supports a distribution $T \neq 0$ whose Fourier transform $\hat{T}$ belongs to L^∞ and vanishes at infinity. This is plainly not the language of Riemann and Cantor; a more traditional definition is adopted in [7], and the equivalence of the two theories is explained in [**5**, V]. The following theorem describes a stronger property of a *closed set* than that deduced from Theorem 4, but does not contain Theorem 4, because it does not concern measures at all.

THEOREM 5. *Let F be a closed set contained in $C \log p_k^{-1}$ intervals of length p_k for a certain sequence $p_k \to 0$. Then F is not an M-set.*

The proof requires knowledge of the work of Pyateckiĭ-Shapiro [7, I, p. 346], especially the definition of an $H^{(r)}$-set. To obtain this theorem, we observe that F is contained in sets of type F_k, for a sequence of integers k tending to infinity.

Given the constant $C>0$, let $r>C \log 2$ be an integer, and $0<b<\frac{1}{2}$ while $\log b > -rC^{-1}$. The integers j in the interval $[1, C \log k]$ are divided into r nearly equal classes, say $L_1, \cdots, L_r$. The Diophantine inequalities, for $n=1, \cdots, r$,

$$(\mathrm{I}_n) \quad |\xi_n x_j| \leqq b \text{ (modulo 1)} \quad \text{for } j \text{ in } L_n, \qquad |\xi_n| \leqq \tfrac{1}{4}(1-2b)k \equiv \delta k,$$

have solutions ξ_n forming a set Q_n, whose cardinalities tend to infinity. Here of course the inequality $-\log b < rC^{-1}$ is decisive. Compare [5, p. 95].

To apply Pyateckiĭ-Shapiro's method [7, I, p. 346] we must choose special r-tuples $(\xi_1, \cdots, \xi_r)$ from $Q=Q_1 \times \cdots \times Q_r$. More exactly, given any finite union U of linear inequalities of the form

$$|a_1\xi_1 + \cdots + a_r\xi_r| \leqq A, \qquad \max |a_n| \geqq 1,$$

we must choose a point $(\xi_1, \cdots, \xi_r)$ in $Q \sim U$. Now in one of our inequalities, with (say) $|a_r| \geqq 1$, any $\xi_1, \cdots, \xi_{r-1}$ admits at most $2A+1$ choices for ξ_r, so U meets Q in a very small part, that is $|Q \cap U| = o(|Q|)$ so that $Q \sim U$ is nonvoid for large k.

Consequently, a set F is of type $H^{(r)}$ provided it is contained in $C \log k$ intervals of length k^{-1}, for an unbounded sequence of integers k. This conclusion, with $C \log k$ strengthened to $o(\log k)$, is due to Salem [**5**, p. 95]; [**4**] and [**5**] treat slightly different properties, and for these $o(\log k)$ is the critical case.

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