A NEW CHARACTERIZATION OF SEPARABLE GCR-ALGEBRAS
TROND DIGERNES

Abstract. It is shown that a separable C*-algebra \( \mathcal{A} \) is GCR if and only if the set of central projections in its enveloping von Neumann algebra \( \mathcal{B} \) is generated, as a complete Boolean algebra, by the set of open, central projections in \( \mathcal{B} \).

1. Let \( \mathcal{A} \) be a C*-algebra, and \( \mathcal{B} \) its enveloping von Neumann algebra, that is, \( \mathcal{B} = \pi_u(\mathcal{A})'' \), where \( \pi_u \) is the direct sum of all cyclic representations of \( \mathcal{A} \). The representation \( \pi_u \) is faithful, and we may therefore consider \( \mathcal{A} \) as a sub-C*-algebra of \( \mathcal{B} \). To each (nondegenerate) representation \( \pi \) of \( \mathcal{A} \) there corresponds a projection \( E' \in \mathcal{B}' = \pi_u(\mathcal{A})' \) such that \( \pi \) may be identified with the map \( A \in \mathcal{A} \mapsto AE' \in \mathcal{B}E' \) [2, §§5 and 12]. A projection \( E \in \mathcal{B} \) is said to be open if it supports a left ideal in \( \mathcal{A} \); that is, if there is a left ideal \( J \) in \( \mathcal{A} \) such that \( J = BE \), where "-" denotes strong closure [1]. We let \( \mathcal{P}_0 \) denote the set of all central projections in \( \mathcal{B} \), \( \mathcal{P}_0 \) the set of open projections in \( \mathcal{P} \) and \( (\mathcal{P}_0) \) the Boolean algebra generated by \( \mathcal{P}_0 \) in \( \mathcal{P} \). With these notations the following has been proved by H. Halpern and the author [5]:

1. \( \mathcal{A} \) is CCR if and only if \( \mathcal{P}_0 \) is strongly dense in \( \mathcal{P} \).
2. If \( \mathcal{A} \) is GCR, then \( (\mathcal{P}_0) \) is strongly dense in \( \mathcal{P} \).

The purpose of this paper is to obtain a converse to 2, at least in the separable case.

For the general theory of C*-algebras and von Neumann algebras we refer the reader to the two books of Dixmier ([2], [3]), especially §§4, 5 and 12 of [2].

2. With notations as above we have:

**Theorem.** For a separable C*-algebra \( \mathcal{A} \) the following two conditions are equivalent:

(i) \( \mathcal{A} \) is GCR;
(ii) \( (\mathcal{P}_0) \) is strongly dense in \( \mathcal{P} \).

**Proof.** (i) \( \Rightarrow \) (ii). See [5].
(ii) $\Rightarrow$ (i). To prove this we use the following characterization of separable GCR algebras, due to Glimm: $\mathcal{A}$ is GCR if and only if any two irreducible representations of $\mathcal{A}$ with the same kernel are equivalent [4].

So let $\pi_1$, $\pi_2$ be irreducible representations of $\mathcal{A}$ with $\ker \pi_1 = \ker \pi_2$, and let $Q_1$, $Q_2$ be the central supports of the minimal projections in $\mathcal{B}' = {}^\pi_\iota(\mathcal{A})'$ corresponding to $\pi_1$ and $\pi_2$ respectively. (The central support $C_E$ of a projection $E$ in a von Neumann algebra $\mathcal{B}$ is defined by $C_E = \inf \{P \in \mathcal{P}; PE = E\}$.) Then $Q_1$ and $Q_2$ are minimal in $\mathcal{P}$. It suffices to show that $Q_1 = Q_2$. We argue by contradiction: Suppose $Q_1 \neq Q_2$; then $Q_1 Q_2 = 0$, by minimality. Let $\mathcal{P}_c$ denote the set of closed, central projections, i.e. $\mathcal{P}_c = \{I - P; P \in \mathcal{P}_0\}$ and set $\mathcal{P}_c = \mathcal{P}_0 \cup \mathcal{P}_c$.

Claim. There is a $P \in \mathcal{P}_c$ such that $Q_1 \leq P$ and $Q_2 \leq I - P$.

Assume, for a moment, this has been proved, and, for definiteness, let $P$ be open. Then there is an ideal $J$ in $\mathcal{A}$ such that $J = BP$, and consequently there is an $A \in J$ with $AQ_1 \neq 0$, since $0 \neq Q_1 \leq P$. On the other hand, $AQ_2 = AP \cdot Q_2(I - P) = AQ_2(I - P) = 0$, contradicting our assumption that $\ker \pi_1 = \ker \pi_2$, and we are through.

So it remains only to prove the Claim. Again we argue by contradiction: Assume there are distinct, minimal projections $Q_1$ and $Q_2$ in $\mathcal{P}$ such that,

\[ (*) \text{ for all } P \in \mathcal{P}_c, (I - P)Q_1 \neq 0 \text{ or } PQ_2 \neq 0. \]

Let $Q = Q_1 + Q_2$ and consider the set:

\[ \mathcal{P}(Q) = \{P \in \mathcal{P}; PQ = Q \text{ or } PQ = 0\}. \]

By $(\ast)$ and by minimality of $Q_1$ and $Q_2$, $\mathcal{P} \subseteq \mathcal{P}(Q)$; and by minimality of $Q_1$ and $Q_2$ again, $\mathcal{P}(Q)$ is closed under finite unions, finite intersections and complementation. It follows that $\langle \mathcal{P}_0 \rangle = \langle \mathcal{P}_c \rangle \subseteq \mathcal{P}(Q)$. Now, by assumption there is a net $\{P_\alpha\}$ from $\langle \mathcal{P}_0 \rangle$ such that $P_\alpha \rightarrow Q_1$ strongly, and, by minimality of $Q_1$, we may assume $P_\alpha \geq Q_1$ for all $\alpha$. But then, since $\langle \mathcal{P}_0 \rangle \subseteq \mathcal{P}(Q)$, also $P_\alpha \geq Q_1 + Q_2$ for all $\alpha$, and consequently $Q_1 = \lim P_\alpha \geq Q_1 + Q_2$, contradiction.

This completes the proof of the theorem.

3. Remark. In the course of the proof we have also established the following: If $\mathcal{A}$ is a $C^*$-algebra (separable or not) with the property that $\langle \mathcal{P}_0 \rangle$ is dense in $\mathcal{P}$, then any two factor-representations of $\mathcal{A}$ with the same kernel are quasi-equivalent.

References


Department of Mathematics, University of California, Los Angeles, California 90024