SELECTION OF REPRESENTING MEASURES FOR INNER PARTS

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Abstract. If a compact convex set $K$ has an inner part $\Delta$ then there is a selection of pairwise boundedly absolutely continuous representing measures for $\Delta$ if and only if $K$ is finite dimensional.

Let $K$ denote a compact convex set in a LCTVS, $A(K)$ the affine continuous real functions on $K$, $\mathcal{P}(K)$ the set of regular Borel probability measures on $K$. Let $\Phi: \mathcal{P}(K) \to K$ be the map which associates to each measure $\mu$ its barycentre. Then $\Phi$ is affine, weak* continuous, and onto $K$. If $\Phi(\mu) = x$ we say $\mu$ represents $x$.

If $L$ is any convex set, $x, y \in L$ and $r > 0$, we say $[x, y]$ extends by $r$ in $L$ if $x + r(x - y) \in L$ and $y + r(y - x) \in L$. We write $x \sim y$ if $\exists r > 0$ such that $[x, y]$ extends by $r$ in $L$. This is an equivalence relation on $L$ and the equivalence classes are the parts of $L$. It is easy to show that $\Phi$ carries parts into parts: If $\Pi$ is a part of $\mathcal{P}(K)$ then $\Phi(\Pi)$ is contained in a part of $K$. Conversely if $\Delta$ is a part of $K$ and $F$ is any finite subset of $\Delta$ then there exists a part $\Pi$ of $\mathcal{P}(K)$ such that $F \subset \Phi(\Pi)$. Indeed if $F = \{x_1, x_2, \ldots, x_n\}$ choose $y_i$ and $z_i$ in $K$ such that $x_i \in (y_i, z_i)$, the open line segment with endpoints $y_i$ and $z_i$, and $x_i \in (y_i, z_i)$ ($2 \leq i \leq n$). If $\Phi(\mu_i) = y_i$ and $\Phi(\nu_i) = z_i$ for $\mu_i, \nu_i \in \mathcal{P}(K)$, then the part $\Pi$ containing $\sum (\mu_i + \nu_i)/(2n - 2)$ satisfies $F \subset \Phi(\Pi)$. Indeed since $x_i \in (y_i, z_i)$ for each $i$, we can clearly find a representing measure $\omega$ for $x_i$ in $\Pi$. Since $x_i \in (y_i, z_i)$, an affine combination of $\mu_i$ and $\omega$ yields a representing measure for $x_i$ in $\Pi$.

Thus if $\Delta$ is a part of $K$ one might ask whether

\begin{equation}
\Delta = \Phi(\Pi) \quad \text{for some part } \Pi \text{ of } \mathcal{P}(K).
\end{equation}

Indeed Bear posed this question in [3] and reproduced an example of Har'kova [4] to show that (1) need not hold if $\mathcal{P}(K)$ is replaced by $\mathcal{P}(\Gamma)$ where $\Gamma$ is the Shilov boundary of $A(K)$.

Since two probability measures $\mu$ and $\nu$ on $K$ are in the same part of $\mathcal{P}(K)$ if and only if $\mu \leq k\nu$ and $\nu \leq k\mu$ for some $k$, condition (1) asserts

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the existence for $\Delta$ of a selection of representing measures on $K$ which are pairwise boundedly absolutely continuous. There are two special cases when (1) is true for all parts $\Delta$ of $K$. One is when $K$ is a simplex, for then there are unique maximal representing measures [6, §9], the other when $K$ is finite dimensional (Theorem 1).

Let $K' = \{x \in K : (\forall y \in K)(\exists r > 0)x + r(x - y) \in K\}$. It can happen that $K' = \emptyset$, but if $K' \neq \emptyset$ it is a part of $K$ called the inner part. Finite dimensional convex sets, for example, always have nonempty inner parts. In Theorem 1 we show that if $\Delta = K' \neq \emptyset$ then (1) holds for $\Delta$ if and only if $K$ is finite dimensional.

First some preliminaries. If $L$ is a compact convex set, $x, y \in L$ and $x \sim y$; let

$$d(x, y) = \inf\{\log(1 + 1/r) : [x, y] \text{ extends by } r\}.$$ 

In [3, Lemma 3.4] it is shown that $d$ is a metric on each part of $L$, called the part metric. Now denote by $d$ and $D$ the part metrics on $K$ and $\mathcal{P}(K)$ respectively and let

$$b(x, r) = \{y \in K : d(x, y) \leq r\} \quad \text{and} \quad B(\mu, r) = \{\nu \in \mathcal{P}(K) : D(\mu, \nu) \leq r\}.$$ 

**Lemma 1.** Suppose $A$ is a part of $K$, $U$ a part of $\mathcal{P}(K)$, and $A = \bigcup_{i=1}^{n} \mathcal{B}(v, r)$. Then there exist $\mu \in \Pi$ and positive numbers $M$ and $k$ such that if $x = \Phi(\mu)$ then

$$b(x, \log(1 + 1/M)) \subset \Phi(B(\mu, \log k)).$$

**Proof.** If $\nu \in \Pi$ then the sets $\Phi(B(\nu, r))$ are closed in the part metric topology. Indeed suppose $x_n = \Phi(\mu_n)$ with $\mu_n \in B(\nu, r)$ and $d(x_n, x) \to 0$. Choose a subset $\mu_n$ converging weak* to $\mu$. Since $B(\nu, r)$ is weak* closed (easy to check), $\mu \in B(\nu, r)$. Since $\Phi$ is weak* continuous, $x_n$ converges in $K$ to $\Phi(\mu)$. But since $d(x_n, x) \to 0$, $x_n$ converges in $K$ to $x$, hence $x = \Phi(\mu) \in \Phi(B(\nu, r))$. (It is an easily verified general fact that in any part of a compact convex set the part metric topology is stronger than the relativized compact topology.)

Since $\Delta = \Phi(\Pi) = \bigcup_{n=1}^{\infty} \Phi(B(\nu, n))$ and the part metric on $\Delta$ is complete [1, §3], the Baire category theorem tells us that we can find $x \in \Delta$ and integers $h$ and $M$ such that $b(x, \log(1 + 1/M)) \subset \Phi(B(\nu, h))$. Choose $\mu \in \Pi$ such that $\Phi(\mu) = x$ and choose $k$ such that $B(\nu, h) \subset B(\mu, \log k)$. \square

**Lemma 2.** Suppose $x \in K^i$. Then $\exists \delta > 0$ such that

$$y \in K \Rightarrow x + \delta(x - y) \in K.$$ 

**Proof.** Let $H = K - x$. Then $0 \in H^i$ and so $H \cap -H$ is closed, convex and absorbs each point of $H$ and $-H$. Since $H$ is compact, convex,
$H \cap -H$ absorbs $H$ \cite[Corollary 10.2]{5}. Thus $\exists \delta > 0$ such that $\delta H \subset H \cap -H \subset -H$. Thus

$$y \in K \Rightarrow y - x \in H \Rightarrow \delta(y - x) \in -H \Rightarrow x + \delta(x - y) \in K. \quad \square$$

If $A$ is a normed linear space and $\epsilon \geq 0$ let $B_\epsilon = \{ h \in A : \|h\| \leq \epsilon \}$.

**Lemma 3.** Suppose $E$ is a normed linear space and $G$ is a weak* closed subspace of the dual space $E^*$. Suppose $x \in E^*$, $r \geq 0$ and $(x + B_r) \cap G = \emptyset$. Then $\exists f \in E$ such that $\|f\| = 1$, $f(G) = 0$ and $f(x) > r$.

**Proof.** $x + B_\epsilon$ is weak* compact and $G$ is weak* closed. Hence $\exists f \in E$ such that $\|f\| = 1$, $f(G) < \alpha$ and $f(x + B_r) \supseteq \alpha$ for some $\alpha$. Since $G$ is a subspace, $\alpha > 0$ and $f(G) = 0$. Since $\|f\| = 1$ we can find $y \in B_\epsilon$ such that $f(y) > r - \alpha$. Then $x - y \in x + B_r$ so $f(x - y) \geq \alpha$ hence $f(x) \geq \alpha + f(y) > r$. \quad \square

Now for the main theorem. We always think of $K$ as embedded in the Banach space $A(K)^*$ with the weak* topology. The norm of $A(K)^*$ provides a metric topology on $K$ which we will refer to as the norm topology.

**Theorem 1.** Suppose $\Delta = K^i \neq \emptyset$. Then the following are equivalent.

1. $A = \Phi(\Pi)$ for some part $\Pi$ of $\mathcal{P}(K)$.
2. $K$ is finite dimensional.

**Proof.** ($1 \Rightarrow 2$). Suppose ($1$) and suppose that $K$ is metrizable. We will show that, in this case, $K$ is finite dimensional. Then we will reduce the general case to this one.

We first show that $K$ is norm separable. If $\mu \in \Pi$ then $\Pi \subset L^1(\mu)$ (via Radon Nikodym), and the norm topology that $\Pi$ gets from $L^1(\mu)$ is the same as the norm topology it gets as a subset of $C(K)^*$. Indeed if $g, h \in L^1(\mu)$ then

$$\sup_{f \in C(K), \|f\| = 1} \int f(g - h) \, d\mu = \|g - h\|_1,$$

where $\|\cdot\|$ denotes the variation norm in the Banach space $\mathcal{M}(K)$ of Radon measures on $K$. Since $L^1(\mu)$ is separable ($K$ is metrizable), $\Pi$ is norm separable in $C(K)^*$. Since $\Phi$ is the restriction to $\mathcal{P}(K)$ of the natural, norm-decreasing surjection $\Phi : C(K)^* \rightarrow A(K)^*$, $\Delta = \Phi(\Pi)$ is norm separable. Since $\Delta = K^i$ is norm dense in $K$, $K$ is norm separable.

Now we show that $K$ is norm compact. Since $K$ is norm complete it will be enough to find for any $\epsilon > 0$ a finite set $F \subset A(K)^*$ such that $K \subset F + B_\epsilon$. So suppose $\epsilon > 0$. Choose $\mu$, $M$, $k$ and $x$ from Lemma 1 and $\delta$ from
Lemma 2 so that $\delta(1+1/M) \leq 1$. Since $K$ is norm separable, we can cover $K$ with countably many balls of norm radius $r = \varepsilon/2M$. A finite number of these balls contains all but at most $\gamma = \varepsilon/2kM$ of the measure $\mu$. Let $P$ be a finite dimensional subspace of $A(K)^*$ containing $x$ and the centres of these finitely many balls.

We claim that $K \subseteq P + B_x$. Indeed suppose $y \in K$ but $y \notin P + B_x$. Let $z = x + (\delta/M)(y - x)$. Then $z \in K$ and $d(x, z) \leq \log(1+1/M)$. Indeed $x + M(x - z) = x + \delta(x - y)$ which is in $K$ by Lemma 2, and $z + M(z - x) = x + \delta(1+1/M)(y - x)$ which is in $K$ since $\delta(1+1/M) \leq 1$. So by Lemma 1 we can choose $v \in B(\mu, \log k)$ such that $z = \Phi(v)$. An easy computation shows that $dv = g d\mu$ with $1/k \leq g \leq k$. Also, since $P$ is weak* closed and $y \notin P + B_x$ we can find $f \in A(K)$ such that $\|f\| = 1$, $f(P) = 0$ and $\langle f(y) \rangle > \varepsilon$ (Lemma 3). Then

$$f(z) = (\delta/M)f(y) > \varepsilon\delta/M,$$

and

$$v(f) = \int fg \ d\mu = \int_{|\|f\| \leq r}fg \ d\mu + \int_{|\|f\| > r}fg \ d\mu \leq r \int g \ d\mu + \|f\| \mu(|f| > r) \leq r + k\gamma = \varepsilon\delta/M$$

(where $\mu(|f| > r) \leq \gamma$ since $|f(w)| > r \Rightarrow w \notin P + B_x$). Since $f \in A(K)$ and $\Phi(v) = z$ we must have $v(f) = f(z)$, a contradiction.

So $K \subset P + B_x$. Hence $K = [(K + B_x) \cap P] + B_x$. Now $(K + B_x) \cap P$ is finite dimensional and norm bounded, so relatively norm compact, and we can choose a finite set $F \subset A(K)^*$ so that $F + B_x$ contains it. Hence $K \subset F + B_x$.

So $K$ is norm compact. We deduce that the unit ball $B_1$ of $A(K)^*$ is norm compact. Indeed it follows from the Hahn Banach Theorem that every element of $A(K)^*$ is given by a Radon measure on $K$. Use the Hahn decomposition of this measure and the fact that any probability measure on $K$ has a barycentre in $K$ to deduce that, for any $\lambda \in B_1$, there exists $k, h \in K$ and $0 \leq \alpha, \beta \leq 1$ such that

$$\lambda(f) = \alpha f(k) - \beta f(h) \quad (f \in A(K)).$$

Thus $B_1$ is contained in a continuous image of $K \times K \times [0, 1] \times [0, 1]$, and is norm compact. It follows that $A(K)^*$ is finite dimensional, and so is $K$.

Now drop the metrizability assumption; suppose $K$ has (1) but is not finite dimensional. Choose a countably infinite, linearly independent sequence $\{f_n\} \subset A(K)$ such that $\|f_n\| \leq 2^{-n}$. Define the map $\Psi$ from $K$ into $l^2$ by $\Psi(x) = f_n(x)$. $\Psi$ is affine and continuous, hence maps $K$ onto a compact convex subset $H$ of $l^2$. From Lemma 4 below $H^1 = \Psi(K^1) \neq \emptyset$. Since every $x \in K^1$ has a representing measure in $\Pi$, every $h \in H^1$ has a
representing measure in \( \Pi \circ \Psi^{-1} = (\mu \circ \Psi^{-1}; \mu \in \Pi) \). Since \( \Pi \) is a part of \( \mathcal{P}(K) \), \( \Pi \circ \Psi^{-1} \) is contained in a part of \( \mathcal{P}(H) \) (from linearity of the map \( \mu \rightarrow \mu \circ \Psi^{-1} \)). So \( H \) has property (1) and since it is metrizable it is, by the first part of the proof, finite dimensional. This contradicts the linear independence of \( \{ f_n \} \).

**Lemma 4.** Suppose \( K \) and \( H \) are convex sets and \( K \neq \emptyset \). Suppose \( \Psi: K \rightarrow H \) is affine and onto. Then \( H = \Psi(K) \).

**Proof.** Clearly \( \Psi(K') \subset H' \). Assume \( x \in H' \). Choose \( z' \in K' \) and let \( z = \Psi(z') \). Since \( x \in H' \), \( x = \lambda z + (1 - \lambda)w \) for some \( w \in H \), \( 0 < \lambda < 1 \). Choose \( w' \in K \) such that \( \Psi(w') = w \). Then if \( x' = \lambda z' + (1 - \lambda)w' \) we have \( \Psi(x') = x \) and \( x' \in K' \) since \( z' \in K' \) and \( 0 \leq \lambda < 1 \). So \( x \in \Psi(K) \).

(2)\( \Rightarrow \) (1). Suppose \( K \) is of dimension \( m \) and is in fact contained in \( R^m \). If \( x \in K' \) then \( K \) contains an open line segment containing \( x \) in the direction of each coordinate axis. From the convexity of \( K \) we deduce that \( K \) and hence \( K' \) contains an open ball in \( R^m \) containing \( x \). Hence \( K' \) is open in \( R^m \).

Choose \( \{ z_i \} \), a countable dense subset of \( E(K) \). Let \( \mu = \sum_i \delta(z_i)2^{-i} \) (\( \delta(z) \) = delta measure at \( z \)). We will show \( K' \subset \Phi(K) \) where \( \Pi \) is the part of \( \mathcal{P}(K) \) containing \( \mu \). Choose \( y \in K' \). Let \( \Phi(\mu) = x \in K \). Since \( y \in K' \) we can choose \( w \in K \) and \( 1 > \alpha > 0 \) so \( y = \alpha x + (1 - \alpha)w \). Choose \( \varepsilon > 0 \) so, \( \forall g \in R^m, ||g - w|| < \varepsilon \Rightarrow g \in K \) (\( ||\cdot|| \) is Euclidean norm in \( R^m \)).

Choose \( n \) so \( \{ z_1, z_2, \cdots, z_n \} \) is an \( \varepsilon \)-net for \( E(K) \). We claim that \( w \in \text{co}\{ z_1, z_2, \cdots, z_n \} \). If not \( \exists \gamma \in R^m, \| \gamma \| = 1 \) such that \( (\gamma, w) \neq (\gamma, z_i) \) for \( 1 \leq i \leq n \). Now \( w + \varepsilon\gamma \in K \). Thus \( \exists z \in E(K) \) so that

\[
(\gamma, z) \geq (\gamma, w + \varepsilon\gamma) = (\gamma, w) + \varepsilon > (\gamma, z_i) + \varepsilon, \quad 1 \leq i \leq n.
\]

It follows that \( \| z - z_i \| > \varepsilon \) if \( 1 \leq i \leq n \). This contradicts the choice of \( n \).

So \( w \in \text{co}\{ z_1, z_2, \cdots, z_n \} \). This provides a measure \( \nu \in \mathcal{P}(K) \) such that \( \Phi(\nu) = w \) and \( \nu \leq 2^n \mu \). Clearly the probability measure \( \alpha \mu + (1 - \alpha)\nu \) represents \( y \). It is in \( \Pi \) since \( \alpha > 0 \) and \( \alpha \mu \leq \alpha \mu + (1 - \alpha)\nu \leq (\alpha + 2^n) \mu \).

**Remarks.** (1) I am grateful to H. S. Bear for his interest in this work. He pointed out to me that my original proof of Theorem 1 was valid only for metrizable \( K \), and supplied the simple geometric proof of Lemma 4. I am also grateful to the referee for indicating several places where a few more details would substantially improve the exposition.

(2) A stronger version of Lemma 1 follows immediately from Bauer’s open mapping theorem (to appear in Equationes Mathematicae, see [3, Theorems 5–13]).
(3) There remains the problem for general parts: Find a condition (geometric or topological) on a part $\Delta$ of $K$ equivalent to (1).

REFERENCES


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