PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, No. 2, December 1972

ON CERTAIN CONVOLUTION INEQUALITIES

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ABSTRACT. It is proved that certain convolution inequalities are easy consequences of the Hardy-Littlewood-Wiener maximal theorem. These inequalities include the Hardy-Littlewood-Sobolev inequality for fractional integrals, its extension by Trudinger, and an interpolation inequality by Adams and Meyers. We also improve a recent extension of Trudinger's inequality due to Strichartz.

The purpose of this note is to point out that certain convolution inequalities are easy consequences of the Hardy-Littlewood-Wiener maximal theorem. These inequalities include the Hardy-Littlewood-Sobolev inequality for fractional integrals, its extension by Trudinger [**11**], and an interpolation inequality by Adams and Meyers ([**1**], [1a]). We also improve a recent extension of Trudinger's inequality due to Strichartz [**10**].

Let f be real-valued, Lebesgue measurable, and defined in $\boldsymbol{R}^d$. For $0<p<\infty$ we write $\|f\|_p=\{\int_{\boldsymbol{R}^d}|f(x)|^p\,dx\}^{1/p}$. For $0<\alpha<d$ the Riesz potentials $I_\alpha(f)$ are defined by $I_\alpha(f)(x)=\int_{\boldsymbol{R}^d} f(y)\,|x-y|^{\alpha-d}\,dy$. The maximal function $M(f)$ is defined by $M(f)(x)=\sup_{r>0} r^{-d}\int_{|y|<r}|f(x+y)|\,dy$. We will denote various constants, independent of f, by A.

By the Hardy-Littlewood-Wiener maximal theorem, a simple proof of which is given by Stein [**9**, I.1],

$$\|M(f)\|_p \leqq A\|f\|_p, \quad p>1. \tag{1}$$

If f is supported by a finite ball B, then (see [**9**, I.5.2])

$$\int_B M(f)\,dx \leqq A\int_B (1+|f|\log^+|f|)\,dx. \tag{2}$$

The following theorem is due to Hardy and Littlewood [**3**] for $d=1$, and to Sobolev [**8**] in the general case. A simple proof is given in [**9**, V.1.2].

THEOREM 1. *Let* $0<\alpha<d$, $1<p<q<\infty$, *and* $1/q=1/p-\alpha/d$. *Then* $\|I_\alpha(f)\|_q\leqq A\|f\|_p$. *If f is supported by a ball B, and* $1/q=1-\alpha/d$, *then* $I_\alpha(f)\in L^q(B)$ *if* $\int_B|f|\log^+|f|\,dx<\infty$.

We first prove a simple lemma.

Received by the editors February 18, 1972 and, in revised form, May 2, 1972.
AMS 1970 *subject classifications*. Primary 26A33, 46E30, 46E35.

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LEMMA. (a) *If* $0<\alpha<d$, *then for all* $x\in \boldsymbol{R}^d$ *and* $\delta>0$

$$\int_{|y-x|\leqq\delta} |f(y)|\,|x-y|^{\alpha-d}\,dy \leqq A\,\delta^\alpha M(f)(x).$$

(b) *If* $\beta>0$, *then*

$$\int_{|y-x|\geqq\delta} |f(y)|\,|x-y|^{-\beta-d}\,dy \leqq A\,\delta^{-\beta} M(f)(x).$$

PROOF. We only prove (a), the proof of (b) being similar. For any $x\in\boldsymbol{R}^d$ and any $\delta>0$

$$\begin{aligned}\int_{|y-x|\leqq\delta} |f(y)|\,|x-y|^{\alpha-d}\,dy &= \sum_{n=0}^{\infty}\int_{\delta 2^{-n-1}<|y-x|\leqq\delta 2^{-n}} |f(y)|\,|x-y|^{\alpha-d}\,dy\\ &\leqq A\sum_{n=0}^{\infty}(\delta 2^{-n})^\alpha(\delta 2^{-n})^{-d}\int_{|y-x|\leqq\delta 2^{-n}}|f(y)|\,dy\\ &\leqq A\,\delta^\alpha M(f)(x)\sum_{n=0}^{\infty}2^{-n\alpha},\end{aligned}$$

which proves (a).

PROOF OF THEOREM 1. Let $p\geqq 1$, $0<\alpha<d$, and let $x\in\boldsymbol{R}^d$ and $\delta>0$ be arbitrary. By Hölder's inequality for $p>1$, and immediately for $p=1$,

$$\int_{|y-x|>\delta} |f(y)|\,|x-y|^{\alpha-d}\,dy \leqq A\,\|f\|_p\,\delta^{\alpha-d/p}.$$

Thus, by the lemma

$$|I_\alpha(f)(x)| \leqq A(\delta^\alpha M(f)(x) + \delta^{\alpha-d/p}\|f\|_p).$$

To minimize this expression we choose $\delta=(M(f)(x)/\|f\|_p)^{-p/d}$. This gives

$$|I_\alpha(f)(x)| \leqq AM(f)(x)^{1-\alpha p/d}\,\|f\|_p^{\alpha p/d}. \tag{3}$$

The theorem follows immediately from (1) and (2).

REMARK. The main difference between the above proof and the proof given in [**9**] is that the latter depends on the nondiagonal case of the Marcinkiewicz interpolation theorem, whereas the proof of the maximal theorem, as given e.g. in [**9**], only depends on the easier interpolation along the diagonal, and a covering lemma. On the other hand the proof in [**9**] is valid for more general kernels.

THEOREM 2. *Suppose* $f\in L^p(\boldsymbol{R}^d)$ *has support in a ball* B *with diameter* R, *and let* $p=d/\alpha>1$. *Then, for any* $\varepsilon>0$ *there exists a constant* A_ε, *independent of f and R, such that*

$$\int_B \exp\left\{\frac{d}{\omega_{d-1}}\left|\frac{|I_\alpha(f)(x)|}{\|f\|_p} - \varepsilon\right|^{p/(p-1)}\right\} dx \leqq A_\varepsilon R^d.$$

Here ω_{d-1} is the area of the $d-1$ dimensional unit sphere.

REMARK. The above inequality clearly implies that for any $\beta < d/\omega_{d-1}$ there is an A so that

$$\int_B \exp \beta(|I_\alpha(f)(x)|/\|f\|_p)^{p/(p-1)}\, dx \leqq AR^d.$$

For $\alpha=1$ this implies the inequality of Trudinger [**11**, p. 479]. In fact, if φ belongs to the Sobolev space $\overset{\circ}{W}{}_1^p(B)$, then $|\varphi| \leqq \omega_{d-1}^{-1} I_1(|\text{grad}\ \varphi|)$. (See e.g. [**9**, V.2.2].) It is known ([**4**], [**6**]) that for $\alpha=1$ the inequality does not hold for $\beta > d/\omega_{d-1}$, but Moser [**6**] has proved that in this case the inequality is true for $\beta = d/\omega_{d-1}$, i.e. ε above can be taken to be zero. The extension of Trudinger's inequality to $\alpha \neq 1$ is due to Strichartz [**10**], whose proof is also very simple, but does not seem to give quite as sharp a result.

PROOF. Without loss of generality we can assume that $\|f\|_p = 1$. As in the proof of Theorem 1 we obtain for any $x \in B$ and any δ, $0 < \delta \leqq R$,

$$|I_\alpha(f)(x)| \leqq A\ \delta^\alpha M(f)(x) + (\omega_{d-1} \log(R/\delta))^{1-1/p}.$$

Now choose $\delta^\alpha = \min\{\varepsilon A^{-1} M(f)(x)^{-1}, R^\alpha\}$. This gives

$$|I_\alpha(f)(x)| \leqq \varepsilon + (\omega_{d-1} \log^+(R\varepsilon^{-1/\alpha} A^{1/\alpha} M(f)(x)^{1/\alpha}))^{1-1/p};$$

$$(|I_\alpha(f)(x)| - \varepsilon)_+^{p/(p-1)} \leqq \omega_{d-1}\, d^{-1} \log^+(R^d \varepsilon^{-p} A^p M(f)(x)^p), \tag{4}$$

and the theorem follows.

The following theorem is a special case of Theorem 4 below, but we prove it separately because of the simplicity of the proof.

THEOREM 3. *Let $f \geqq 0$ be measurable on $\boldsymbol{R}^d$. Then*

$$\|I_{\alpha\theta}(f)\|_r \leqq A\ \|f\|_p^{1-\theta}\ \|I_\alpha(f)\|_q^\theta,$$

with $0<\alpha<d$, $0<\theta<1$, $1<p<\infty$, $p<q\leqq\infty$, *and* $1/r=(1-\theta)/p+\theta/q$.

REMARK. For periodic functions, and without the restriction to positive functions, the theorem was proved by Hirschman [**5**]. For integral α and $\alpha\theta$ the theorem follows from the well-known inequalities of Gagliardo [**2**] and Nirenberg [7]. See also Theorem 4 below.

PROOF. Let x be arbitrary and let $\delta > 0$. Then clearly

$$\begin{aligned}\int_{|y-x|\geqq\delta} f(y)\,|x-y|^{\alpha\theta-d}\,dy &\leqq \delta^{\alpha(\theta-1)} \int_{|y-x|\geqq\delta} f(y)\,|x-y|^{\alpha-d}\,dy \\ &\leqq \delta^{\alpha(\theta-1)} I_\alpha(f(x)).\end{aligned}$$

Thus, by the lemma $I_{\alpha\theta}(f)(x) \leqq A(\delta^{\alpha\theta}M(f)(x)+\delta^{\alpha(\theta-1)}I_\alpha(f)(x))$. Choose $\delta^\alpha = I_\alpha(f)(x)/M(f)(x)$. It follows that

$$I_{\alpha\theta}(f)(x) \leqq AM(f)(x)^{1-\theta} I_\alpha(f)(x)^\theta. \tag{5}$$

The theorem follows immediately from Hölder's inequality and (1).

The following theorem was recently announced by Adams and Meyers [**1**]. Their proof, which is by complex interpolation, will appear in [**1a**].

THEOREM 4. *Let $f \geqq 0$ be measurable on $\boldsymbol{R}^d$. Then*

$$\|I_{\alpha\theta}(f^t)\|_r \leqq A\, \|f\|_p^{t-\theta}\, \|I_\alpha(f)\|_q^\theta,$$

with $0<\alpha<d$, $0<\theta<1$, $0<p<\infty$, $0<q\leqq\infty$, $\theta<t<\theta+(1-\theta)p$, *and* $1/r=(t-\theta)/p+\theta/q$.

PROOF. The case $t=1$ was treated above. We first assume $t>1$. Then, by the assumptions, $t<p$. As before, by the lemma,

$$\int_{|y-x|\leqq\delta} f^t(y)\,|x-y|^{\alpha\theta-d}\,dy \leqq A\delta^{\alpha\theta}M(f^t)(x).$$

Now choose $s<p$ so that $t<\theta+(1-\theta)s$. By Hölder's inequality

$$\begin{aligned} &\int_{|y-x|\geqq\delta} f^t(y)\,|x-y|^{\alpha\theta-d}\,dy \\ &\quad\leqq \left\{\int_{|y-x|\geqq\delta} f(y)\,|x-y|^{\alpha-d}\,dy\right\}^{(s-t)/(s-1)} \\ &\quad\quad\cdot\left\{\int_{|y-x|\geqq\delta} f^s(y)\,|x-y|^{-d-\alpha\eta/(t-1)}\,dy\right\}^{(t-1)/(s-1)}, \end{aligned} \tag{6}$$

where $\eta=\theta-t+(1-\theta)s>0$.

By part (b) of the lemma

$$\int_{|y-x|\geqq\delta} f^s(y)\,|x-y|^{-d-\alpha\eta/(t-1)}\,dy \leqq A\delta^{-\alpha\eta/(t-1)}M(f^s)(x).$$

We now observe that $M(f^t)^{1/t} \leqq AM(f^s)^{1/s}$ for $0<t\leqq s$. In fact for all $a>0$, by Hölder's inequality,

$$\left\{a^{-d}\int_{|y-x|<a} f^t(y)\,dy\right\}^{1/t} \leqq A\left\{a^{-d}\int_{|y-x|<a} f^s(y)\,dy\right\}^{1/s} \leqq AM(f^s)(x)^{1/s}.$$

Thus

$$I_{\alpha\theta}(f^t)(x) \leqq A\delta^{\alpha\theta}M(f^s)(x)^{t/s} + A\delta^{-\alpha\eta/(s-1)}M(f^s)(x)^{(t-1)/(s-1)}I_\alpha(f)(x)^{(s-t)/(s-1)}.$$

To minimize this expression we choose $\delta^\alpha = I_\alpha(f)(x)/M(f^s)(x)^{1/s}$. It follows that

$$I_{\alpha\theta}(f^t)(x) \leqq AM(f^s)(x)^{(t-\theta)/s} I_\alpha(f)(x)^\theta. \tag{7}$$

Since $\|M(f^s)^{1/s}\|_p \leqq A\|f\|_p$ by (1), the theorem follows by Hölder's inequality.

Now assume $t<1$. We choose s_1 and s_2, so that $0<s_2<s_1<p$, $s_1 \leqq t$, and so that $\theta+(1-\theta)s_2<t<\theta+(1-\theta)s_1$. By the assumptions this is possible.

Applying Hölder's inequality as in (6) we find

$$\int_{|y-x|\leqq\delta} f^t(y)\,|x-y|^{\alpha\theta-d}\,dy \leqq \left\{\int_{|y-x|\leqq\delta} f(y)\,|x-y|^{\alpha-d}\,dy\right\}^{(t-s_1)/(1-s_1)} \cdot \left\{\int_{|y-x|\leqq\delta} f^{s_1}(y)\,|x-y|^{-d+\alpha\eta_1/(1-t)}\,dy\right\}^{(1-t)/(1-s_1)},$$

where $\eta_1=\theta-t+(1-\theta)s_1>0$.

Similarly

$$\int_{|y-x|\geqq\delta} f^t(y)\,|x-y|^{\alpha\theta-d}\,dy \leqq \left\{\int_{|y-x|\geqq\delta} f(y)\,|x-y|^{\alpha-d}\,dy\right\}^{(t-s_2)/(1-s_2)} \cdot \left\{\int_{|y-x|\geqq\delta} f^{s_2}(y)\,|x-y|^{-d+\alpha\eta_2/(1-t)}\,dy\right\}^{(1-t)/(1-s_2)},$$

where $\eta_2=\theta-t+(1-\theta)s_2<0$.

By the lemma

$$\int_{|y-x|\leqq\delta} f^{s_1}(y)\,|x-y|^{-d+\alpha\eta_1/(1-t)}\,dy \leqq A\delta^{\alpha\eta_1/(1-t)}M(f^{s_1})(x),$$

and

$$\begin{aligned}\int_{|y-x|\geqq\delta} f^{s_2}(y)\,|x-y|^{-d+\alpha\eta_2/(1-t)}\,dy &\leqq A\delta^{\alpha\eta_2/(1-t)}M(f^{s_2})(x)\\ &\leqq A\delta^{\alpha\eta_2/(1-t)}M(f^{s_1})(x)^{s_2/s_1}.\end{aligned}$$

Thus

$$\begin{aligned}I_{\alpha\theta}(f^t)(x) &\leqq A\delta^{\alpha\eta_1/(1-s_1)}M(f^{s_1})(x)^{(1-t)/(1-s_1)}I_\alpha(f)(x)^{(t-s_1)/(1-s_1)}\\ &\quad + A\delta^{\alpha\eta_2/(1-s_2)}M(f^{s_1})(x)^{s_2(1-t)/s_1(1-s_2)}I_\alpha(f)(x)^{(t-s_2)/(1-s_2)}\\ &= A\delta^{\alpha\theta}\{\delta^{\alpha(s_1-t)}M(f^{s_1})(x)^{1-t}I_\alpha(f)(x)^{t-s_1}\}^{1/(1-s_1)}\\ &\quad + A\delta^{\alpha\theta}\{\delta^{\alpha(s_2-t)}M(f^{s_1})(x)^{s_2(1-t)/s_1}I_\alpha(f)(x)^{t-s_2}\}^{1/(1-s_2)}.\end{aligned}$$

By choosing $\delta^\alpha=I_\alpha(f)(x)/M(f^{s_1})(x)^{1/s_1}$, we find

$$I_{\alpha\theta}(f^t(x)) \leqq AM(f^{s_1})(x)^{(t-\theta)/s_1}I_\alpha(f)(x)^\theta, \tag{8}$$

which proves the theorem.

Note that in the case $t<p$, $t<1$, we can simplify the proof by choosing $s_1=t$.

REFERENCES

1. D. R. Adams and N. G. Meyers, *Bessel potentials. Inclusion relations among classes of exceptional sets*, Bull. Amer. Math. Soc. **77** (1971), 968–970.

1a. ———, *Bessel potentials. Inclusion relations among classes of exceptional sets* (to appear).

2. E. Gagliardo, *Ulteriori proprietà di alcune classi di funzioni in più variabili*, Ricerche Mat. **8** (1959), 24–51. MR **22** #181.

3. G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals*. I, Math. Z. **27** (1928), 565–606.

4. J. A. Hempel, G. R. Morris and N. S. Trudinger, *On the sharpness of a limiting case of the Sobolev imbedding theorem*, Bull. Austral. Math. Soc. **3** (1970), 369–373.

5. I. I. Hirschman, Jr., *A convexity theorem for certain groups of transformations*, J. Analyse Math. **2** (1953), 209–218. MR **15**, 295; 1139.

6. J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

7. L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 115–162. MR **22** #823.

8. S. L. Sobolev, *On a theorem of functional analysis*, Mat. Sb. **4 (46)** (1938), 471–497; English transl., Amer. Math. Soc. Transl. (2) **34** (1963), 39–68.

9. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton, Univ. Press, Princeton, N.J., 1970.

10. R. S. Strichartz, *A note on Trudinger's extension of Sobolev's inequalities*, Indiana Univ. Math. J. **21** (1972), 841–842.

11. N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483. MR **35** #7121.

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