THE SUBCONTINUA OF A DENDRON FORM
A HILBERT CUBE FACTOR

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Abstract. The title statement is proved, and it is shown further that the subcontinua of a dendron actually form a Hilbert cube when (and only when) the branch points of the dendron are dense. Along the way, it is established that whenever a Hilbert cube manifold is compactified into a Hilbert cube factor by the addition of another Hilbert cube factor having Property Z in the compactification, then the resulting space is actually a Hilbert cube.

It is known ([1], [14]) that the Cartesian product of any dendron with the Hilbert cube is homeomorphic to the Hilbert cube. In this paper, the hyperspace $C(D)$ of all nonvoid subcontinua of a dendron $D$ is investigated and found to have the same property. In fact, it is shown that if the branch points of $D$ are dense, then $C(D)$ is a Hilbert cube.

The method used presents in simple form the techniques employed elsewhere by the author and R. M. Schori ([11], [12]) to show that the hyperspace of all nonvoid, closed subsets of the unit interval is a Hilbert cube, and Theorem 1 is applied crucially in the extension of that result to finite, connected graphs in [13].

The space $X$ will be termed a Hilbert cube factor if for some space $Y$, $X \times Y$ is homeomorphic to the Hilbert cube, $Q$. It is easy to see that this condition holds if and only if $X \times Q$ is a Hilbert cube. (See [15, Lemma 2].)

A dendron $D$ is a uniquely arcwise connected Peano continuum. The branch points of $D$ are those points which separate it into more than two...
components. Furthermore, each dendron $D$ may be written as the closure in the plane of a countable union of arcs $a_i$, of lengths converging to zero, such that

$$a_i \cap \left( T_{i-1} = \bigcup_{j=1}^{i-1} a_j \right) = a_i,$$

is an endpoint of $a_i$ [16]. This, together with the unique arcwise connectivity, allows the dendron to be written as the inv lim{$T_i, r_i$}, where $r_i: T_i \to T_{i-1}$ is defined by retracting $a_i$ to $a_i$. (The $r_i$'s extend to retractions of $D$.) Finally, a dendron always admits a convex metric $\rho$, i.e., one for which there always exists a point halfway between any two given points. The metric $\rho$ may always be chosen so that $b$ separates $a$ from $c$ in $D$ if and only if the points $a$, $b$, and $c$ are distinct and $\rho(a, c) = \rho(a, b) + \rho(b, c)$. (See [14] for a simple construction of such a metric by embedding in the Hilbert cube.)

We are concerned with the hyperspace $C(D)$ of all (nonvoid) subcontinua of $D$ under the topology generated by the Hausdorff metric $d_H$, which is defined by the condition that $d_H(K_1, K_2) < \varepsilon$ if and only if each point of $K_1$ is within $\varepsilon$ of some point of $K_2$ and vice versa. The inverse limit expression $\text{inv lim}\{T_i, r_i\}$ of $D$ given above induces one $\text{inv lim}\{C(T_i), r_i\}$ of $C(D)$ when $r_i$ is the map $C(T_i) \to C(T_{i-1})$ induced by $r_i$. We shall show that this expression guarantees that $C(D)$ is a Hilbert cube factor, but first it is necessary to introduce some notation.

A closed subset $A$ of a separable, metric ANR $X$ has Property Z in $X$ if and only if for every open set $U$ of $X$ the inclusion map $U \setminus A \to U$ is a homotopy equivalence. (This is the appropriate generalization to ANR's of R. D. Anderson's definition [2] of Property Z in infinite-dimensional manifolds. See, for example, the article by Eells and Kuiper [9], in which it is shown that $A$ has Property Z in $X$ if for every $a \in A$ there are arbitrarily small open neighborhoods $U$ of $a$ in $X$ with the property that each map of a sphere into $U$ may be deformed in $U$ to a map into $U \setminus A$, and each map $f: (B^n, S^{n-1}) \to (U, U \setminus A)$ is homotopic in $U$ to a map $g: B^n \to U \setminus A$ by a homotopy which is constant on $S^{n-1} \times I$. See also [4]. This condition is guaranteed whenever there is a homotopy $H: X \times I \to X$ with $H(x, 0) = x$ for each $x \in X$ which has the property that $H(X \times (0, 1]) \cap A = \varnothing$.) The importance of Property Z in the Hilbert cube is expressed by Anderson's homogeneity theorem [2, Theorem 10.1]: Each homeomorphism between two subsets of the Hilbert cube with Property Z extends to a homeomorphism of the Hilbert cube. This theorem will be used immediately to establish Theorem 1 below, which gives a sufficient condition for the compactification of a Hilbert cube manifold to be a Hilbert cube.
THEOREM 1. Let $X$ be a compactification of a Hilbert cube manifold $M$ such that both $X$ and the remainder, $A = X\setminus M$, are Hilbert cube factors. If $A$ has Property Z in $X$, then $X$ is a Hilbert cube.

Proof. The pair $(X, A)$ is a pair of Hilbert cube factors, so $(X \times Q, A \times Q)$ is a pair of Hilbert cubes. Moreover, $A \times Q$ has Property Z in $X \times Q$, as is easily seen. The homogeneity theorem then allows a homeomorphism of pairs $(X \times Q, A \times Q) \to (A \times Q \times I, A \times Q \times \{1\})$ extending the natural homeomorphism on the second entries. Thus, the space $Y$ obtained from $X \times Q$ by simultaneously identifying $\{a\} \times Q$ to a point $a$ for each $a$ in $A$ is homeomorphic to the result, $Z$, of identifying each $\{a\} \times Q \times \{1\}$ to a point in $A \times Q \times I$. However, $Z$ is just the product of $A$ with the cone $C(Q)$ of $Q$. Note that $C(Q)$ is a Hilbert cube: One way of seeing this is to observe that $Q$, hence $C(Q)$, may be embedded in Hilbert space as a compact, convex, infinite-dimensional set, so the theorem of O. H. Keller, [10] that all such are Hilbert cubes applies. Since $C(Q)$ is a Hilbert cube and $A$ is a Hilbert cube factor, $Z$ is also a Hilbert cube, and so is $Y$. There remains only to show that $Y$ is homeomorphic to $X$, which is done as follows: $Y \setminus A$ is homeomorphic to $X \setminus A$ by [3], because it is homeomorphic to $X \times Q \setminus A \times Q = M \times Q$ and each Hilbert cube manifold is homeomorphic to its product with a Hilbert cube. Also, $Y \setminus A$, the result of identifying $A$ to a point, is a Hilbert cube because $Y \setminus A$ is homeomorphic to $Z \setminus A = C(A \times Q)$ which is homeomorphic to a Hilbert cube. Hence, $Y \setminus A$, and therefore $M$, embeds in $Q$ as an open subset. Now, the proof of the theorem in the addendum of [3] shows that for any open cover $\mathcal{U}$ of $M$ there is a homeomorphism $h: M \times Q \to M$ which is a $\mathcal{U}$-close to the projection $p: M \times Q \to M$, i.e., the cover $\{(p(x), h(x)) | x \in M \times Q\}$ refines $\mathcal{U}$. The proper choice of $\mathcal{U}$ yields a homeomorphism of $Y \setminus A$ onto $M$ extending to a homeomorphism of $Y$ onto $X$ which is the identity on $A$. Since $Y$ is a Hilbert cube, so is $X$.

The following is the main theorem of this note.

THEOREM 2. The subcontinua $C(D)$ of a dendron $D$ form a Hilbert cube factor which is a Hilbert cube if (and only if) the branch points of $D$ are dense.

Proof. Write $D = \text{inv lim} \{T_i, r_i\}$ and for each $i > 1$ let $b_i$ be the endpoint of the arc $\alpha_i = \text{cl}(T_i \setminus T_{i-1})$ which does not lie in $T_{i-1}$. Now consider

$$r'_i = \tilde{r}_i | C_{b_i}(T_i): C_{b_i}(T_i) \to C(T_{i-1}),$$

where $C_{b_i}(T_i)$ denotes those members of $C(T_i)$ containing $b_i$. Now let $M_{r'_i}$ be the mapping cylinder of $r'_i$, that is,

$$M_{r'_i} = [C_{b_i}(T_i) \times I \cup C(T_{i-1})]/\Xi,$$
where $\Xi$ is the upper-semicontinuous decomposition with nondegenerate elements all sets of the form \( \{K\} \cup (r_{i-1}^{-1}(K) \times \{0\}) \) where $K$ is in $C_u(T_{i-1})$.

There is a homeomorphism $g_i$ of $C(T_i)$ onto $M_{r_i'}$ for which the following diagram commutes, $c_i$ being the collapse of $M_{r_i'}$ to $C(T_{i-1})$ given by $[K, t] \mapsto r_i'(K)$ if $K$ is in $C_u(T_i)$ and $[K] \mapsto K$ if $K$ is in $C(T_{i-1})$.

\[
\begin{array}{ccc}
C(T_i) & \xrightarrow{g_i} & M_{r_i'} \\
\downarrow{\hat{r}_i} & & \downarrow{c_i} \\
C(T_{i-1}) & & 
\end{array}
\]

This $g_i$ may be defined by parameterizing $a_i$ with $[0, 1]$ so that $a_i \mapsto 0$ and $b_i \mapsto 1$ and then letting

\[
g_i(K) = \begin{cases} [K], & \text{if } K \cap a_i = \emptyset, \\ [K \cup a_i, d], & \text{if } a_i \in K, \\ [(1/d)K, d], & \text{if } K \subseteq a_i \setminus \{a_i\}, \end{cases}
\]

where $d = \sup(K \cap a_i)$ if $K \cap a_i \neq \emptyset$.

Both $C(T_i)$ and $C_u(T_i)$ are polyhedra by [6] and [8] (see also [7] for a definitive analysis), and they are easily seen to be contractible because $T_i$ is contractible to $b_i$. Therefore, by [14] they are Hilbert cube factors, so by [15],

\[
c_i \times \text{id}: M_{r_i'} \times Q \rightarrow C(T_{i-1}) \times Q
\]

is a uniform limit of homeomorphisms. The commutative diagram above shows that

\[
\hat{r}_i \times \text{id}: C(T_i) \times Q \rightarrow C(T_{i-1}) \times Q
\]

is also a uniform limit of homeomorphisms. Thus,

\[
\text{inv lim}\{C(T_i) \times Q, \hat{r}_i \times \text{id}\}
\]

being an inverse limit of Hilbert cubes and uniform limits of homeomorphisms, is a Hilbert cube by [5]. However, $\text{inv lim}\{C(T_i) \times Q, \hat{r}_i \times \text{id}\}$ is easily seen to be homeomorphic to $\text{inv lim}\{C(T_i), \hat{r}_i\} \times Q = C(D) \times Q$, so $C(D)$ is a Hilbert cube factor.

If now, the branch points of $D$ are not dense, it is immediate that $C(D)$ contains an open 2-cell, namely, those nondegenerate members which lie entirely within some open arc which is an open subset of $D$. Therefore, in this case $C(D)$ is not itself a Hilbert cube.

On the other hand, if the branch points of $D$ are dense, Theorem 1 may be employed to show $C(D)$ a Hilbert cube by identifying $D$ with the set
of all degenerate subcontinua of itself and showing that (1) it has Property Z in \( C(D) \), and (2) \( C(D) \setminus D \) is a Hilbert cube manifold. \( D \) itself is a Hilbert cube factor as may be seen by using the proof already given for \( C(D) \) or by citing [14]. The fact was originally proven by R. D. Anderson [1], but his proof was never published.)

It is easy to show that \( D \) has Property Z in \( C(D) \), because, using the metric \( \rho \) selected at the outset, the homotopy \( H : C(D) \times I \to C(D) \) sending \( (K, t) \) to the closed \( t \)-neighborhood of \( K \) in \( D \) satisfies the homotopy condition mentioned parenthetically which guarantees Property Z. There remains, then, only to verify that \( C(D) \setminus D \) is a Hilbert cube manifold. Again using the metric \( \rho \), it is easy to see that for any nondegenerate subcontinuum \( K \) of \( D \) there are an \( \varepsilon > 0 \) and two other nondegenerate subcontinua \( K_- \) and \( K_+ \) of \( D \) with the property that the closed \( \varepsilon \)-neighborhood \( N \) of \( K \) in \( C(D) \) is the set of all members of \( C(D) \) containing \( K_- \) yet lying in \( K_+ \). Setting \( H = K_+ / K_- \), it is easy to see that the map \( C(K_-) \to C(K_+) \) induced from \( K_+ \to K_- \) carries \( N \) homeomorphically to \( C_*(H) \), where \( * = K_+/K_- \). Also, because the branch points of \( D \) are dense, the branch points of \( K \) lying in \( K_- \) are dense in \( K_+ \), and thus * separates \( H \) into infinitely many components with closures \( H_i, i = 1, 2, \ldots \). Now, the intersection maps \( \bigcap H_i : C_*(H) \to C_*(H_i) \) are continuous, and their product \( \bigcap : C_*(H) \to \prod_{i=1}^{\infty} C_*(H_i) \) is a homeomorphism. Furthermore, the proof that \( C(D) \) is a Hilbert cube factor will also apply without modification to show that each \( C_*(H_i) \) is one also. Therefore, \( N \) is homeomorphic to a countably infinite product of nondegenerate Hilbert cube factors and by [14] is itself a Hilbert cube. This shows that \( C(D) \setminus D \) is a Hilbert cube manifold, so Theorem 1 applies to show that \( C(D) \) is a Hilbert cube.

REFERENCES


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