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ON A CLASS OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. We look at functions $f(z)$ for which there correspond functions $\phi(z)$ convex of order α such that $\operatorname{Re}\{f'(z)/\phi'(z)\} \geqq \beta$. We examine the influence of the second coefficient of $\phi(z)$ on this class. In particular, distortion, covering, and radius of convexity theorems are proved.

1. **Introduction.** Let S be the class of normalized univalent functions analytic in the unit disk E ($|z|<1$). Let $K_p(\alpha)$ denote the subclass of S consisting of functions of the form $\phi(z)=z+b_2z^2+\cdots$, where

$$\operatorname{Re}\{z\phi''(z)/\phi'(z)+1\} \geqq \alpha, \qquad z \in E, \ |b_2| = p,$$

and $0 \leqq \alpha \leqq 1$. This class is called convex of order α. It is known that $0 \leqq p \leqq 1$. In addition, $\phi(z) \in K_p(1)$ if and only if $\phi(z)=z$ and $\phi(z) \in K_1(\alpha)$ if and only if $\phi(z)=z/(1-xz)$, $|x|=1$.

We say that an analytic function $f(z)=z+a_2z^2+\cdots$ is in the class $C_p(\alpha, \beta)$ if there exists a function $\phi(z) \in K_p(\alpha)$ such that

$$\operatorname{Re}\{f'(z)/\phi'(z)\} \geqq \beta, \qquad \beta \geqq 0.$$

It is easy to verify that for $\alpha \leqq \alpha_0$ and $\beta \leqq \beta_0$ we have

$$C_p(\alpha_0, \beta) \subset C_p(\alpha, \beta) \quad \text{and} \quad C_p(\alpha, \beta_0) \subset C_p(\alpha, \beta).$$

Kaplan [5] proved that $C_p(0, 0)$, the class of close-to-convex functions, is univalent. Hence $C_p(\alpha, \beta)$ is a subclass of S.

By specializing α and β we obtain some important classes. If $f(z)$ is in $C_p(0, 0)$, then $f(z)$ is close-to-convex; $C_p(1, \beta)$, then $\operatorname{Re} f'(z) \geqq \beta$; $C_p(\alpha, 1)$, then $f(z)$ is convex of order α; $C_p(1, 1)$, then $f(z)=z$.

In this note we prove distortion, covering, and radius of convexity theorems for the class $C_p(\alpha, \beta)$. By specializing p, some of our results will coincide with those of Libera [6]. We also look at a corresponding subclass of the close-to-star functions, $S_p(\alpha, \beta)$. Some of our results for this class will generalize those of Al-Amiri [1] who investigated the class $S_p(\alpha, 0)$.

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In the sequel, we will assume that $f(z)$ is in $C_p(\alpha, \beta)$ with $\phi(z)$ its associated function in $K_p(\alpha)$.

2. **Distortion theorems for** $C_p(\alpha, \beta)$**.** We begin by proving an existence theorem for functions in this class.

THEOREM 1. *Let* $\alpha \in [0, 1]$, $\beta \in [0, 1]$, *and* $p \in [0, 1-\alpha]$. *Then there exists a function* $f(z) \in C_p(\alpha, \beta)$. *This result is sharp in that* $\alpha+p \leqq 1$ *for any* α.

In proving the theorem we will make use of the following

LEMMA. *Let* $Q(z)$ *be analytic for* $z \in E$ *with* $Q(0)=1$. *Then* $\operatorname{Re} Q(z) \geqq \beta$ *if and only if*

$$Q(z) = \frac{1 + (1 - 2\beta)g(z)}{1 - g(z)},$$

where $g(z)$ *is analytic*, $g(0)=0$, *and* $|g(z)|<1$ *for* $z \in E$.

PROOF OF LEMMA. The result is well known for $\alpha=0$. In the general case, let $Q(z)=(1-\alpha)P(z)+\alpha$, where $P(z)$ satisfies the conditions of the Lemma with $\alpha=0$.

PROOF OF THEOREM 1. The inequality $\alpha+p\leqq 1$ is proved in [**1**, p. 104]. Let

$$f(z) = \int_0^z \left(\frac{1}{1 - t^2}\right)^{1-\alpha} \left(\frac{1 + t}{1 - t}\right)^p \left(\frac{1 + (1 - 2\beta)t}{1 - t}\right) dt$$

and

$$\phi(z) = \int_0^z \left(\frac{1}{1 - t^2}\right)^{1-\alpha} \left(\frac{1 + t}{1 - t}\right)^p dt.$$

Since

$$\frac{f'(z)}{\phi'(z)} = \frac{1 + (1 - 2\beta)z}{1 - z}$$

has real part $\geqq \beta$, it suffices to show that $\phi(z) \in K_p(\alpha)$.

We have

$$1 + z\frac{\phi''(z)}{\phi'(z)} = \frac{1 + 2pz + (1 - 2\alpha)z^2}{1 - z^2} = \frac{1 + (1 - 2\alpha)g(z)}{1 - g(z)}.$$

Solving for $g(z)$, we obtain

$$g(z) = z\left(\frac{z + p/(1 - \alpha)}{1 + (p/(1 - \alpha))z}\right) = zh(z).$$

Since $\alpha+p\leqq 1$, $h(z)$ maps $E \to E$, and $|g(z)| \leqq |z| < 1$ for $z \in E$. Since $g(z)$ satisfies the conditions of the Lemma, our proof is complete.

THEOREM 2. *Let* $f(z) \in C_p(\alpha, \beta)$. *Then*

$$(1) \qquad |f'(z)| \leqq \left(\frac{1}{1-r^2}\right)^{1-\alpha} \left(\frac{1+r}{1-r}\right)^p \left(\frac{1+(1-2\beta)r}{1-r}\right),$$

$$(2) \qquad |f'(z)| \geqq \frac{1}{(1+(2p/(1-\alpha))r+r^2)^{1-\alpha}} \left(\frac{1-(1-2\beta)r}{1+r}\right),$$

where the first expression on the right-hand side of (2) *is taken to be* 1 *for* $\alpha=1$.

Equality holds in (1) *for the function*

$$f_1(z) = \int_0^z \left(\frac{1}{1-t^2}\right)^{1-\alpha} \left(\frac{1+t}{1-t}\right)^p \left(\frac{1+(1-2\beta)t}{1-t}\right) dt$$

and equality holds in (2) *for the function*

$$f_2(z) = \int_0^z \frac{1}{(1+(2p/(1-\alpha))t+t^2)^{1-\alpha}} \left(\frac{1-(1-2\beta)t}{1+t}\right) dt.$$

PROOF. From the Lemma we obtain

$$(3) \qquad \frac{f'(z)}{\phi'(z)} = \frac{1+(1-2\beta)g(z)}{1-g(z)},$$

where $g(0)=0$ and $|g(z)|<1$ for $z \in E$.

Since $g(z)$ satisfies the conditions of Schwarz's lemma, (3) yields

$$(4) \qquad \frac{1-(1-2\beta)r}{1+r} \leqq \left|\frac{f'(z)}{\phi'(z)}\right| \leqq \frac{1+(1-2\beta)r}{1-r}.$$

In [**1**, p. 105] it is proved that

$$(5) \qquad \frac{1}{(1+(2p/(1-\alpha))r+r^2)^{1-\alpha}} \leqq |\varphi'(z)| \leqq \left(\frac{1}{1-r^2}\right)^{1-\alpha} \left(\frac{1+r}{1-r}\right)^p.$$

Combining (4) and (5), the result follows. In the proof of Theorem 1 it was shown that $f_1(z) \in C_p(\alpha, \beta)$. The proof that $f_2(z) \in C_p(\alpha, \beta)$ is similar, with

$$\phi_2(z) = \int_0^z \frac{1}{(1+(2p/(1-\alpha))t+t^2)^{1-\alpha}}\, dt.$$

REMARK. For $p=1-\alpha$, (1) reduces to a result of Libera and (2) improves on a result of Libera [**6**, p. 152]. In his paper, it is claimed that the function

$$f(z)=\int_0^z \frac{1-t}{(1+t)^{2(1-\alpha)}(1+t(1-2\beta))}\,dt$$

is in $C_{1-\alpha}(\alpha, \beta)$ for every α and β. That this is not the case can be seen by letting $\alpha=1$ and $\beta=\frac{1}{2}$. Then $f(z)=z-z^2/2$ and $\phi(z)=z$. But

$$\operatorname{Re}\{f'(z)/\phi'(z)\} = \operatorname{Re}\{1-z\}$$

which is less than $\frac{1}{2}$ for $\frac{1}{2}<z<1$, and $f(z)=z-z^2/2 \notin C_{1-\alpha}(1, \frac{1}{2})$.

THEOREM 3. *Let $f(z) \in C_p(\alpha, \beta)$. Then*

$$\int_0^r \frac{1}{(1+(2p/(1-\alpha))t+t^2)^{1-\alpha}}\left(\frac{1-(1-2\beta)t}{1+t}\right) dt \leqq |f(z)|$$

$$\leqq \int_0^r \left(\frac{1}{1-t^2}\right)^{1-\alpha}\left(\frac{1+t}{1-t}\right)^p \left(\frac{1-(1-2\beta)t}{1+t}\right) dt.$$

Equality holds on the right-hand side for $f_1(z)$ in Theorem 2 and on the left-hand side for $f_2(z)$ in Theorem 2.

PROOF. Integrating along the straight line segment from the origin to $z=re^{i\theta}$ and applying Theorem 2 we obtain

$$|f(z)| \leqq \int_0^r |f'(te^{i\theta})|\, dt \leqq \int_0^r \left(\frac{1}{1-t^2}\right)^{1-\alpha}\left(\frac{1+t}{1-t}\right)^p \left(\frac{1-(1-2\beta)t}{1+t}\right) dt,$$

which proves the right-hand inequality. To prove the left-hand inequality, for every r we choose z_0, $|z_0|=r$, such that

$$|f(z_0)| = \min_{|z|=r} |f(z)|.$$

If $L(z_0)$ is the pre-image of the segment $\{0, f(z_0)\}$, then

$$|f(z)| \geqq |f(z_0)| \geqq \int_{L(z_0)} |f'(z)|\,|dz|$$

$$\geqq \int_0^r \frac{1}{(1+(2p/(1-\alpha))t+t^2)^{1-\alpha}}\left(\frac{1-(1-2\beta)t}{1+t}\right) dt$$

and this completes the proof.

For $p=1-\alpha$, the right-hand inequality reduces to a result of Libera and the left-hand inequality sharpens a result of Libera [**6**, p. 152].

3. **Covering theorems for $C_p(\alpha, \beta)$.** We first prove a coefficient theorem for the class.

THEOREM 4. *Let* $f(z) \in C_p(\alpha, \beta)$, *with* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$. *Then* $|a_2| \leqq 1+p-\beta$, *with extremal function*

$$f(z) = \int_0^z \left(\frac{1}{1-t^2}\right)^{1-\alpha} \left(\frac{1+t}{1-t}\right)^p \left(\frac{1+(1-2\beta)t}{1-t}\right) dt.$$

PROOF. Let $\phi(z)=z+\sum_{n=2}^{\infty} b_n z^n$. Then

$$g(z) = \frac{f'(z)/\varphi'(z) - \beta}{1-\beta} = 1 + \frac{2(a_2 - b_2)}{1-\beta} + \sum_{n=2}^{\infty} c_n z^n$$

has positive real part in E. Hence [3, p. 15], $2|a_2-b_2|/(1-\beta) \leqq 2$, or $|a_2| \leqq 1+|b_2|-\beta=1+p-\beta$.

THEOREM 5. *Let* $f(z) \in C_p(\alpha, \beta)$, *with* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$. *If* $f(z) \neq k$ *for* $z \in E$, *then* $|k| \geqq 1/(3+p-\beta)$.

PROOF. If $f(z)$ does not assume the value k, then

$$\frac{kf(z)}{k - f(z)} = z + (a_2 + 1/k)z^2 + \sum_{n=3}^{\infty} c_n z^n$$

is in the class S. Hence,

$$|a_2 + 1/k| \leqq 2. \tag{6}$$

Applying the triangle inequality and Theorem 4 to (6) we obtain Theorem 5.

Since $p \leqq 1-\alpha$, we also obtain the following result of Libera [6, p. 155] as a

COROLLARY. $|k| \geqq 1/(4-\alpha-\beta)$.

4. **A radius of convexity theorem for $C_p(\alpha, \beta)$.**

THEOREM 6. *Let* $f(z) \in K_p(\alpha, \beta)$. *Then* $f(z)$ *maps the disk* $|z|<R$ *onto a convex domain, where* R *is the least positive root of the equation* $a(r, p, \alpha, \beta)=0$, *where*

$$\begin{aligned} a(r,p,\alpha,\beta) = {} & (1-\alpha)(1-2\alpha)(1-2\beta)r^4 \\ & - 2[(1-\alpha)(1-\beta) + \alpha p(1-2\beta) \\ & \qquad - \beta(1-\alpha)(1-2\alpha)]r^3 \\ & - 2[(1-\alpha)(1-\alpha-\beta-2p\beta) + 2p]r^2 \\ & - 2[1-\alpha(1+p)]r + (1-\alpha). \end{aligned}$$

PROOF. Let $f'(z)/\phi'(z)=Q(z)$, where Re $Q(z) \geqq \beta$. Since

$$z\frac{f''(z)}{f'(z)} = z\frac{\phi''(z)}{\phi'(z)} + z\frac{Q'(z)}{Q(z)}, \tag{7}$$

the radius of convexity of $f(z)$ is at least equal to the smallest positive root of

$$1 + \min \operatorname{Re}\{z\phi''(z)/\phi'(z)\} + \min \operatorname{Re}\{zQ'(z)/Q(z)\} = 0.$$

In [**1**, p. 105], it is shown that

$$\operatorname{Re}\left\{z\frac{\phi''(z)}{\phi'(z)}\right\} \geqq -\frac{2r(1-\alpha)[r(1-\alpha)+p]}{(1-\alpha)(1+r^2)+2pr}. \tag{8}$$

Now let $Q(z)=(1-\beta)P(z)+\beta$, where $P(z)$ is analytic, $P(0)=1$, and $\operatorname{Re} P(z)>0$ in E. Then

$$z\frac{Q'(z)}{Q(z)} = \frac{(1-\beta)P'(z)}{(1-\beta)P(z)+\beta} = \frac{P'(z)}{P(z)+\beta/(1-\beta)}.$$

Using a lemma of Libera [**6**, p. 150] we obtain

$$\left|z\frac{Q'(z)}{Q(z)}\right| \leqq \frac{2r}{(1-r)[1+r+(\beta/(1-\beta))(1-r)]}. \tag{9}$$

Substituting (8) and (9) into (7) yields

$$\begin{aligned}\operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} &\geqq 1-\frac{2r(1-\alpha)[r(1-\alpha)+p]}{(1-\alpha)(1+r^2)+2pr} \\ &\quad -\frac{2r}{(1-r)[1+r+(\beta/(1-\beta))(1-r)]}.\end{aligned} \tag{10}$$

Simplifying the right-hand side of (10) we obtain

$$\frac{a(r,p,\alpha,\beta)}{[(1-\alpha)(1+r^2)+2pr][(1-r)(1+r+(\beta/(1-\beta))(1-r)]},$$

and this completes the proof.

REMARK 1. The Koebe function is in $C_1(0,0)$, and the least positive root of

$$a(r,1,0,0) = r^4-2r^3-6r^2-2r+1$$

is $2-\sqrt{3}$, the radius of convexity for the class S.

REMARK 2. If $p=1-\alpha$,

$$\begin{aligned} a(r,&1-\alpha,\alpha,\beta) \\ &= (r+1)(1-\alpha)[(1-2\alpha)(1-2\beta)r^3 \\ &\qquad -(3-6\beta+4\alpha\beta)r^2+(2\alpha-3)r+1]. \end{aligned}$$

This reduces to a result of Libera [**6**, p. 151].

5. **The class** $S_p(\alpha, \beta)$. Let $S_p^*(\alpha)$ denote the class of functions analytic in E and of the form $s(z)=z+a_2z^2+\cdots$, where $s(z)$ is starlike of order α and $|a_2|=2p$.

Let $S_p(\alpha, \beta)$ denote the class of functions analytic in E and of the form $g(z)=z+b_2z^2+\cdots$ such that $\operatorname{Re}\{g(z)/s(z)\}\geqq\beta$ for $z \in E$ and for some $s(z) \in S_p^*(\alpha)$.

The class $S_p(0, 0)$, defined by Reade [7], is called close-to-star. It is known that members of this class need not be univalent. However, there is an important connection between the classes $S_p(\alpha, \beta)$ and $C_p(\alpha, \beta)$ which we state as

THEOREM 7. *The following relationships hold*:

$$f(z) \in C_p(\alpha, \beta) \quad \textit{if and only if} \quad zf'(z) \in S_p(\alpha, \beta), \tag{11}$$

$$f(z) \in S_p(\alpha, \beta) \quad \textit{if and only if} \quad \int_0^z \frac{f(t)}{t}\, dt \in C_p(\alpha, \beta). \tag{12}$$

PROOF. It is well known that $\phi(z)$ is convex of order α if and only if $z\phi'(z)$ is starlike of order α. Hence, $\phi(z)=z+a_2z^2+\cdots$ is in $K_p(\alpha)$ if and only if $z\phi'(z)=z+2a_2z^2+\cdots$ is in $S_p^*(\alpha)$. Since

$$\operatorname{Re}\{f'(z)/\phi'(z)\} = \operatorname{Re}\{zf'(z)/z\phi'(z)\},$$

we obtain (11). The proof of (12) is similar and will be omitted.

THEOREM 8. *Let* $g(z) \in S_p(\alpha, \beta)$. *Then*

$$\frac{r}{(1 + (2p/(1-\alpha))r + r^2)^{1-\alpha}}\left(\frac{1-(1-2\beta)r}{1+r}\right) \leqq |g(z)| \leqq \frac{r}{(1-r^2)^{1-\alpha}}\left(\frac{1+r}{1-r}\right)^p\left(\frac{1+(1-2\beta)r}{1-r}\right).$$

PROOF. The result follows on combining Theorem 2 with Theorem 7.

COROLLARY. *Let* $g(z)$ *be analytic in* E *with* $\operatorname{Re}\{g(z)/z\}>\frac{1}{2}$. *Then*

$$r/(1+r) \leqq |g(z)| \leqq r/(1-r). \tag{13}$$

PROOF. We have $\operatorname{Re}\{g(z)/z\}>\frac{1}{2}$ if and only if $g(z) \in S_0(1, \frac{1}{2})$.

Schild [8, p. 752] proved (13) for the class of functions starlike of order $\frac{1}{2}$, a subclass of $S_0(1, \frac{1}{2})$ [4, p. 472].

For a functional analysis proof of the corollary, see [2, p. 94].

Once again making use of Theorem 7, we see that $g(z) \in S_p(\alpha, \beta)$ if and only if

$$z\frac{g'(z)}{g(z)} = 1 + z\frac{f''(z)}{f'(z)}$$

for some $f(z) \in C_p(\alpha, \beta)$. Hence a radius of convexity theorem in $C_p(\alpha, \beta)$ will correspond to a radius of starlikeness theorem in $S_p(\alpha, \beta)$. From Theorem 6 we now obtain

THEOREM 9. *Let $g(z) \in S_p(\alpha, \beta)$. Then $g(z)$ maps the disk $|z|<R$ onto a starlike domain, where R is the least positive root of $a(r, p, \alpha, \beta)=0$, defined in Theorem* 6.

For $\beta=0$, this reduces to a result of Al-Amiri [**1**, p. 108].

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