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APPROXIMATION BY RATIONAL FUNCTIONS ON RIEMANN SURFACES

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ABSTRACT. In this paper, we show that if $F \in L^p(k, \alpha)$ on Γ where Γ denotes the border of a compact bordered Riemann surface $\bar{R}$, then F can be uniquely written as the sum of a function in $H^p(k, \alpha)$ and a function in $G^p(k, \alpha)$ and moreover that F can be approximated on Γ in L^p norm to within $A/n^{k+\alpha}$ by a sequence of rational functions on the union of $\bar{R}$ with its double.

The purpose of this paper is to generalize some results of the second named author [4] concerning the Hardy classes H^p and their particularizations to the classes $H^p(k, \alpha)$ of functions satisfying (together with certain derivatives) a Lipschitz condition to Riemann surfaces.

Let γ denote the unit circle. We define the class $L^p(k, \alpha)$ $(0<\alpha<1, 1<p<\infty)$ to be the set of functions possessing derivatives up to order k on γ whose kth derivatives satisfy a pth mean integrated Lipschitz condition of order α. $H^p(k, \alpha)$ will denote the subclass of functions belonging to $L^p(k, \alpha)$ which belong to the Hardy class H^p of the interior of γ while $G^p(k, \alpha)$ will denote those functions belonging to $L^p(k, \alpha)$ which are boundary functions of functions belonging to H^p of the exterior of γ and vanishing at ∞. In [4], it is shown that every function of the class $L^p(k, \alpha)$ can be written uniquely as the sum of a function in $H^p(k, \alpha)$ and a function in $G^p(k, \alpha)$. It is this result that we shall generalize to the setting of Riemann surfaces and in addition obtain some results on approximation by rational functions.

Let R denote the interior of a compact bordered Riemann surface $\bar{R}$ with boundary Γ. Suppose f possesses derivatives up to order k on Γ. Let $\Gamma_1, \cdots, \Gamma_n$ denote the connected components of Γ and let χ_i, $i=1, \cdots, n$, denote the uniformizers which map Γ_i onto γ. We shall say that f is of class $L^p(k, \alpha)$ on Γ if $f \circ \chi_i^{-1}$ is of class $L^p(k, \alpha)$ on γ for all i. This definition is independent of the choice of uniformizing variables as a consequence of a theorem of Hardy and Littlewood [3].

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Let $\zeta \in R$ and denote by $g_\zeta(z)$ the Green's function of R with pole at ζ. Let $\hat{R}$ denote the double of R. By the symmetry principle, $g_\zeta(z)$ has a meromorphic extension to $\bar{R} \cup \hat{R}$ which we shall continue to denote by $g_\zeta(z)$. Let $\delta g_\zeta(z) = 2(\partial g_\zeta(z)/\partial z)\, dz$ and let $\partial_{\delta g_\zeta(z)}$ denote the divisor of $\delta g_\zeta(z)$. Define $H^p(\hat{R}, \partial_{\delta g_\zeta(z)}|\hat{R})$ to be the class of meromorphic functions f on $\hat{R}$ which outside of a compact subset of $\hat{R}$ belong to the Hardy class H^p and have the property that $\partial_f + \partial_{\delta g_a} \geqq 0$. Let $H^p(R)$ denote the Hardy class of analytic functions on R whose moduli raised to the pth power possess a harmonic majorant. Let $H^p(k, \alpha)$ denote the class of functions in $H^p(R)$ whose boundary values belong to $L^p(k, \alpha)$. Let $G^p(k, \alpha)$ denote those functions in $H^p(\hat{R}, \partial_{\delta g_\zeta(z)}|\hat{R})$ whose boundary functions belong to $L^p(k, \alpha)$.

We shall now state and prove our main theorem which is a generalization of Theorem 4 in [**4**].

THEOREM 1. *Let $F(z) \in L^p(k, \alpha)$ on Γ. Then on Γ we may write $F(z) \equiv f(z) + g(z)$ where $f(z) \in H^p(k, \alpha)$ and $g(z) \in G^p(k, \alpha)$.*

PROOF. The assumption that $F \in L^p(k, \alpha)$ on Γ implies that F is of class L^p on Γ and hence by a theorem of Heins [**2**, Theorem 8, p. 81] we may write $F(z) \equiv f(z) + g(z)$ where $f \in H^p(R)$ and $g \in H^p(\hat{R}, \partial_{\delta g_\zeta(z)}|\hat{R})$. Furthermore, $\|f\|_p$, $\|g\|_p \leqq C\|F\|_p$ where C is a constant depending only on p. By $\|F\|_p$, we mean

$$\left(\frac{1}{2\pi}\int_\Gamma |F|^p \frac{\partial g_\zeta(z)}{\partial n}\, dS\right)^{1/p}.$$

Thus it follows from these Riesz type inequalities together with Theorem 1 in [**4**] that f and g are of class $L^p(k, \alpha)$ on Γ. Hence f and g belong respectively to the classes $H^p(k, \alpha)$ and $G^p(k, \alpha)$.

Insofar as approximation by rational functions goes, we have the following result.

THEOREM 2. *Under the conditions of Theorem 1, there exist rational functions $R_n(z)$ on $\bar{R} \cup \hat{R}$ satisfying*

$$\|F(z) - R_n(z)\|_p \leqq A/n^{k+\alpha},$$

and conversely if there exist rational functions $R_n(z)$ on $\bar{R} \cup \hat{R}$ which are free from poles on Γ and which satisfy the above inequality, then $F \in L^p(k, \alpha)$ on Γ.

PROOF. Suppose first of all that $F \in L^p(k, \alpha)$. Then by Theorem 1, p. 264 in [**4**], there exist analytic functions $F_n(z)$ in a two-sided neighborhood $\bar{D}$ of Γ which we can and will take to be a system of ring domains satisfying on Γ the inequality

$$\|F(z) - F_n(z)\|_p \leqq A'/n^{k+\alpha}.$$

It follows from a construction given by Behnke and Sommer [**1**, pp. 583–584] that there exists a meromorphic differential $dw(\zeta, z)$ on $\bar{R}\cup\hat{R}$ which for fixed z is a differential in ζ which on $\bar{R}$ is analytic except for a simple pole at z with residue 1 while for fixed ζ, $w(\zeta, z)$ is a meromorphic function on $\bar{R}\cup\hat{R}$ which is free from poles on Γ. By taking $\bar{D}$ so small that for $\zeta\in\bar{D}$, $\zeta\neq z$, $dw(\zeta, z)$ for fixed z is an analytic differential in $\bar{D}$, we have by the residue theorem [**1**, p. 545] that, for $z\in\Gamma$,

$$F_n(z) = \frac{1}{2\pi i}\int_{\partial D} F_n(\zeta)\, dw(\zeta, z).$$

We now approximate the integral by a sum and thus obtain the existence of rational functions on $\bar{R}\cup\hat{R}$ satisfying the inequality

$$\|F_n(z) - R_n(z)\|_p \leqq 1/n^{k+\alpha}.$$

Combining this with the first inequality, we have on taking $A=A'+1$ the desired conclusion. To obtain the converse, we have merely to apply Theorem 1 in [**4**].

It is interesting to observe that if we apply Heins' theorem to the rational functions $R_n(z)$, we obtain the existence of functions $p_n(z)\in H^p(R)$ and $q_n(z)\in H^p(\hat{R}, \partial_{\delta g_\zeta(z)}|\hat{R})$ such that on Γ,

$$\|f(z) - p_n(z)\|_p \leqq A/n^{k+\alpha}, \quad \|g(z) - q_n(z)\|_p \leqq \tilde{A}/n^{k+\alpha}.$$

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