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ON THE LOWER BOUND OF THE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION WITH INFINITE VARIANCE. II

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ABSTRACT. Let N_n be the number of real roots of a random algebraic equation $\sum_{v=0}^{n} \xi_v x^v = 0$ where the ξ_v's are independent random variables with a common characteristic function

$$\exp(-C\,|t|^{\alpha}), \quad \alpha > 1,$$

and C, a positive constant. Then for $n > n_0$,

$$N_n > (\mu \log n)/(\log \log n)$$

outside a set of measure at most

$$\mu'/\{\log((\log n_0)/(\log \log n_0))\}^{\alpha-1}.$$

1.1. **Introduction.** Let N_n be the number of real roots of a random algebraic equation $\sum_0^n \xi_v x^v = 0$, where the coefficients ξ_v's are independent random variables with identical distributions.

Taking the coefficients as normally distributed or uniformly distributed in $[-1, 1]$ or assuming the values $+1$ and -1 with equal probabilities, Littlewood and Offord [3] have shown that $N_n > (\alpha \log n)/(\log \log \log n)$, except for a set of measure at most $A/\log n$, n being sufficiently large.

Taking the coefficients as normally distributed, Evans [1] has shown that there exists an integer n_0 such that for $n > n_0$,

$$N_n > (\beta \log n)/(\log \log n)$$

except for a set of measure at most $(B \log \log n_0)/(\log n_0)$.

Samal [4] has considered the general case when the ξ_v's have expectation zero with variance and third absolute moment nonzero finite. He has shown that $N_n > \varepsilon_n \log n$ where $\varepsilon_n \to 0$ but $\varepsilon_n \log n \to \infty$ as n tends to infinity. The measure of the exceptional set tends to zero as n tends to infinity.

Recently, Samal and Mishra [5] have considered the case when the ξ_v's have a common characteristic function $\exp(-C|t|^{\alpha})$ where C is a positive constant and $\alpha \geqq 1$. They have obtained that

$$N_n > (\mu \log n)/(\log \log n).$$

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Simple calculation will show that their exceptional set is of measure at most $\mu'/(\log\log n)(\log n)^{\alpha-1}$ if $1 \leqq \alpha < 2$ and $(\mu' \log\log n)/(\log n)$ if $\alpha \geqq 2$, n being sufficiently large.

Our object is to show in this case that for $n > n_0$,

$$N_n > (\mu \log n)/(\log\log n)$$

except for a set of measure at most $\mu'/\{\log((\log n_0)/(\log\log n_0))\}^{\alpha-1}$, when $\alpha > 1$.

Our result is true for all $\alpha > 1$, but its importance lies in the range $1 < \alpha < 2$ when the variance is infinite. However, incidentally we find that the corresponding result of Evans [**1**] happens to be a special case of our result when $\alpha = 2$, although our exceptional set is larger than his.

Positive constants shall be denoted by μ's. We shall suppose that any inequality is satisfied when n is large.

Our proofs are modifications of our earlier ones. At some places, we shall indicate only the conclusion arrived at and try to avoid repetition of similar arguments given in the paper cited above.

2.1. THEOREM. *Let $f(x)$ be a polynomial of degree n whose coefficients are independent random variables with a common characteristic function* $\exp(-C|t|^\alpha)$, *where $\alpha > 1$ and C is a positive constant. Then there exists an integer n_0 such that for each $n > n_0$, the number of real roots of the equation $f(x) = 0$ is at least* $(\mu \log n)/(\log\log n)$ *except for a set of measure at most* $\mu'/\{\log((\log n_0)/(\log\log n_0))\}^{\alpha-1}$.

LEMMA 2.1. *If a random variable $\xi(u)$ has characteristic function* $\exp(-C|t|^\alpha)$ *then*

$$\Pr(|\xi(u)| > \varepsilon) < \frac{2^{1+\alpha}C}{1+\alpha} \cdot \frac{1}{\varepsilon^\alpha},$$

for every $\varepsilon > 0$.

For proof see Samal and Mishra [**5**].

2.2. PROOF OF THE THEOREM. Take constants A and B such that $0 < B < 1$ and $A > 1$. Let

$$\lambda_m = m^{1/\alpha} \log m. \tag{2.1}$$

Let M_m, $m = 1, 2, 3, \cdots$, be a sequence of integers defined by

$$M_m = [(2^\alpha Ae\lambda_m^\alpha)/(Bm)] + 1 \tag{2.2}$$

and k be the integer defined by

$$(2k)!\, M_n^{2k} \leqq n < (2k+2)!\, M_n^{2k+2}. \tag{2.3}$$

It follows from this that

$$(\mu_1' \log n)/(\log \log n) < k < (\mu_2' \log n)/(\log \log n).$$

Hence k is large when n is large.

We consider $f = \sum_{v=0}^{n} \xi_v x^v$ at the points

$$x_m = (1 - 1/((2m)!\, M_m^{2m}))^{1/\alpha}, \quad \text{for } m = [\tfrac{1}{2}k] + 1, [\tfrac{1}{2}k] + 2, \cdots, k. \tag{2.4}$$

We have

$$f(x_m) = U_m + R_m \tag{2.5}$$

where

$$U_m = \sum_{(2m-1)!M_m^{2m-1}+1}^{(2m+1)!M_m^{2m+1}} \xi_v x_m^v$$

and

$$R_m = \left(\sum_{0}^{(2m-1)!M_m^{2m-1}} + \sum_{(2m+1)!M_m^{2m+1}+1}^{n} \right) \xi_v x_m^v.$$

Let

$$V_m = \left(\sum_{(2m-1)!M_m^{2m-1}+1}^{(2m+1)!M_m^{2m+1}} \xi_v x_m^{\alpha v} \right)^{1/\alpha}.$$

2.3. The Lemma 2 and Lemma 3 of Samal and Mishra [5] shall be modified in the following manner.

LEMMA 2.2.

$$\left| \sum_{(2m+1)!M_m^{2m+1}+1}^{n} \xi_v x_m^v \right| < \tfrac{1}{2} V_m$$

except for a set of measure at most

$$(2^{1+2\alpha} \cdot C/(1+\alpha))(Ae/B)e^{-(2m-1)M_m}$$

for every sufficiently large m.

LEMMA 2.3.

$$\left| \sum_{0}^{(2m-1)!M_m^{2m-1}} \xi_v x_m^v \right| < \lambda_m \left(\sum_{0}^{(2m-1)!M_m^{2m-1}} x_m^{\alpha v} \right)^{1/\alpha}$$

except for a set of measure at most $2^{1+\alpha} \cdot C/(1+\alpha)\lambda_m^{\alpha}$.

Proofs of these two lemmas will run in the same line as in the paper cited above.

2.4. Thus for every sufficiently large m,

$$|R_m| < \tfrac{1}{2}V_m + \lambda_m\left(\sum_{0}^{(2m-1)!M_m^{2m-1}} x_m^{\alpha v}\right)^{1/\alpha}$$

outside a set of measure at most $\mu_1 e^{-(2m-1)M_m} + \mu_2/\lambda_m^\alpha$. But

$$\begin{aligned}\lambda_m\left(\sum_{0}^{(2m-1)!M_m^{2m-1}} x_m^{\alpha v}\right)^{1/\alpha} &< \lambda_m\{(2m-1)!\,M_m^{2m-1}+1\}^{1/\alpha}\\ &\leqq \lambda_m \cdot 2^{1/\alpha}\{(2m-1)!\,M_m^{2m-1}\}^{1/\alpha}\\ &\leqq \lambda_m \cdot 2^{1/\alpha}\left\{\frac{(Ae/B)V_m^\alpha}{2mM_m}\right\}^{1/\alpha} < \tfrac{1}{2}V_m.\end{aligned}$$

The last step follows from (2.2) and from the relation

$$V_m^\alpha \geqq (B/Ae)(2m)!M_m^{2m}.$$

Since $k\to\infty$ as $n\to\infty$, it follows that when n is sufficiently large $|R_m|<V_m$ for $m=[\tfrac{1}{2}k]+1, [\tfrac{1}{2}k]+2, \cdots, k$ except for a set of measure at most $\mu_1 e^{-(2m-1)M_m}+\mu_2/\lambda_m^\alpha$.

We define E_m and F_m as follows:

$$\begin{aligned}E_m &= \{U_{2m} > V_{2m},\ U_{2m+1} < -V_{2m+1}\},\\ F_m &= \{U_{2m} < -V_{2m},\ U_{2m+1} > V_{2m+1}\}.\end{aligned}$$

It can be shown as in Samal and Mishra [5] that $m(E_m\cup F_m)=\delta>0$, where δ is independent of m.

Let η_m be a random variable which takes value 1 on $E_m\cup F_m$ and zero elsewhere. In other words,

$$\begin{aligned}\eta_m &= 1 \quad \text{with probability } \delta,\\ &= 0 \quad \text{with probability } 1-\delta.\end{aligned}$$

Let ζ_m be defined as follows

$$\begin{aligned}\zeta_m &= 0 \quad \text{if } |R_{2m}| < V_{2m} \quad \text{and} \quad |R_{2m+1}| < V_{2m+1},\\ &= 1 \quad \text{otherwise.}\end{aligned}$$

Let $\xi_m=\eta_m-\eta_m\zeta_m$. If $\xi_m=1$, there is a root of the polynomial in the interval (x_{2m}, x_{2m+1}). Hence the number of roots in the interval (x_{2m_0}, x_{2k+1}) where $m_0=[\tfrac{1}{2}k]+1$ must exceed $\sum_{m=m_0}^k \xi_m$.

2.5. We shall need the strong law of large numbers in the following form.

Let $\eta_1, \eta_2, \cdots$ be a sequence of independent random variables identically *distributed with $V(\eta_i)<1$ for all i; then for each $\varepsilon>0$,*

$$\Pr\left\{\sup_{k\geqq k_0}\left|\frac{1}{k}\sum_{i=1}^{k}\{\eta_i - E(\eta_i)\}\right| \geqq \varepsilon\right\} \leqq \frac{D}{\varepsilon^2 k_0},$$

where D is a positive constant.

Here we have

$$\sum_{m=m_0}^{k}\{\xi_m - E(\eta_m)\} = \sum_{m=m_0}^{k}\{\eta_m - \eta_m\zeta_m - E(\eta_m)\}$$
$$= \sum_{m=m_0}^{k}\{\eta_m - E(\eta_m)\} - \sum_{m=m_0}^{k}\eta_m\zeta_m.$$

Therefore,

$$\left|\sum_{m=m_0}^{k}\{\xi_m - E(\eta_m)\}\right| \leqq \left|\sum_{m=m_0}^{k}\{\eta_m - E(\eta_m)\}\right| + \left|\sum_{m=m_0}^{k}\eta_m\zeta_m\right|$$
$$\leqq \left|\sum_{m=m_0}^{k}\{\eta_m - E(\eta_m)\}\right| + \sum_{m=m_0}^{k}\zeta_m.$$

Since

$$E(\zeta_m) = 1\cdot\Pr(\zeta_m = 1) \leqq \Pr(|R_m| \geqq V_m)$$
$$\leqq \mu_1 e^{-(2m-1)M_m} + \mu_2/\lambda_m^{\alpha},$$

we have $\sum_{m=m_0}^{k}\zeta_m<(k-m_0+1)\varepsilon_1$, outside an exceptional set of measure at most

$$\sum_{m=m_0}^{k}\frac{1}{(k-m_0+1)\varepsilon_1}\left(\mu_1 e^{-(2m-1)M_m} + \frac{\mu_2}{\lambda_m^{\alpha}}\right)$$
$$< \frac{1}{\varepsilon_1}\{\mu_1 e^{-(2m_0-1)M_{m_0}} + \mu_2/x_{m_0}^{\alpha}\}$$
$$< \frac{1}{\varepsilon_1}\{\mu_1/(2m_0-1)M_{m_0} + \mu_2/\lambda_{m_0}^{\alpha}\} < \mu_3/\lambda_{m_0}^{\alpha}.$$

Thus outside an exceptional set of measure at most $\mu_3\sum_{k-m_0+1\geqq k_0}1/\lambda_{m_0}^{\alpha}$ we shall have

$$\sup_{k-m_0+1\geqq k_0}\frac{1}{k-m_0+1}\left|\sum_{m=m_0}^{k}\{\xi_m - E(\eta_m)\}\right|$$
$$\leqq \sup_{k-m_0+1\geqq k_0}\frac{1}{k-m_0+1}\left|\sum_{m=m_0}^{k}\{\eta_m - E(\eta_m)\}\right| + \varepsilon_1.$$

Now, by using the strong law of large numbers

$$\Pr\left\{\sup_{k-m_0+1\geqq k_0}\frac{1}{k-m_0+1}\left|\sum_{m=m_0}^{k}\{\xi_m-E(\eta_m)\}\right|\geqq\varepsilon\right\}$$

$$\leqq \Pr\left\{\sup_{k-m_0+1\geqq k_0}\frac{1}{k-m_0+1}\left|\sum_{m=m_0}^{k}\{\eta_m-E(\eta_m)\}\right|\geqq\varepsilon-\varepsilon_1\right\}$$

$$\leqq\frac{D}{(\varepsilon-\varepsilon_1)^2k_0}.$$

Hence outside a set G_{k_0} where

$$m(G_{k_0})\leqq\frac{\mu_4}{k_0}+\mu_3\sum_{k-m_0+1\geqq k_0}\frac{1}{\lambda^{\alpha}_{m_0}}$$

we have

$$\sup_{k-m_0+1\geqq k_0}\frac{1}{k-m_0+1}\left|\sum_{m=m_0}^{k}\{\xi_m-E(\eta_m)\}\right|<\varepsilon.$$

Therefore

$$\frac{1}{k-m_0+1}\sum_{m=m_0}^{k}\xi_m>\frac{1}{k-m_0+1}\sum_{m=m_0}^{k}E(\eta_m)-\varepsilon$$

for all k such that $k-m_0+1\geqq k_0$.

Since $E(\eta_m)=\delta$, we have

$$N_n>\sum_{m=m_0}^{k}\xi_m>(k-m_0+1)(\delta-\varepsilon)=(k-[\tfrac{1}{2}k])>(\mu\log n)/(\log\log n),$$

for all k such that $k-m_0+1\geqq k_0$.

If k is even, the statements $k-m_0+1\geqq k_0$ and $k\geqq 2k_0$ are equivalent, and if k is odd, the statements $k-m_0+1\geqq k_0$ and $k\geqq 2k_0-1$ are equivalent. So $N_n>(\mu\log n)/(\log\log n)$ for all $k\geqq 2k_0$.

If $n=n_0$ corresponds to $k=2k_0$, then all $n>n_0$ will correspond to $k>2k_0$. Therefore, for all $n>n_0$,

$$N_n>(\mu\log n)/(\log\log n).$$

Since $\alpha>1$,

$$m(G_{k_0})\leqq\frac{\mu_4}{k_0}+\mu_3\sum_{k\geqq 2k_0-1}\frac{1}{\lambda^{\alpha}_{m_0}}$$

$$=\frac{\mu_4}{k_0}+\mu_3\left\{\frac{1}{\lambda^{\alpha}_{k_0}}+2\left(\frac{1}{\lambda^{\alpha}_{k_0+1}}+\frac{1}{\lambda^{\alpha}_{k_0+2}}+\frac{1}{\lambda^{\alpha}_{k_0+3}}+\cdots\right)\right\}$$

$$<\frac{\mu_4}{k_0}+2\mu_3\sum_{k\geqq k_0}\frac{1}{\lambda^{\alpha}_{k}}$$

$$
\begin{aligned}
&= \frac{\mu_4}{k_0} + 2\mu_3 \sum_{k \geq k_0} \frac{1}{k(\log k)^\alpha} \\
&< \mu_5(\log\log n_0/\log n_0) + \mu_6(1/(\log k_0)^{\alpha-1}) \\
&< \mu_5 \frac{\log\log n_0}{\log n_0} + \mu_7 \frac{1}{\{\log((\mu_1' \log n_0)/(\log\log n_0))\}^{\alpha-1}} \\
&< \mu' \Big/ \left\{\log\left(\frac{\log n_0}{\log\log n_0}\right)\right\}^{\alpha-1}.
\end{aligned}
$$

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