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ON THE CONTINUITY OF BEST POLYNOMIAL APPROXIMATIONS

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ABSTRACT. Suppose f is a continuous complex valued function defined on a compact set E in the plane and $p_n(f, E)$ is the polynomial of degree n of best uniform approximation to f on E. If a polynomial q_n of degree n approximates f on E "almost" as well as $p_n(f, E)$, then q_n is "almost" $p_n(f, E)$. Sharp estimates, one for the real and one for the general case, are found for $\|q_n - p_n(f, E)\|_E$ in terms of the quantity $(\|f-q_n\|_E - \|f-p_n(f, E)\|_E)$, where $\|\cdot\|_E$ denotes the uniform norm on E.

1. **Introduction.** For a function f continuous on E, a compact set in the plane, let $\|f\|_E = \max_{z\in E} |f(z)|$. Also, for $n \in Z^+$, let $p_n(f, E)$ denote the polynomial of degree n of best uniform approximation to f on E. A basic question that arises in the theory of best approximation is: If two continuous functions f_1 and f_2 are "close" on E, are their polynomials of best approximation $p_n(f_1, E)$ and $p_n(f_2, E)$ also "close" on E. More precisely, if $\{f_m\}_{m=1}^{\infty}$ is a sequence of continuous functions converging uniformly to f on E, does the sequence $\{p_n(f_m, E)\}_{m=1}^{\infty}$ converge uniformly to $p_n(f, E)$ on E (for each n) and if so, how rapid is the convergence.

The above problem can be stated in even greater generality. Suppose f is continuous on a compact set E, $n \in Z^+$, $p_n(f, E) \equiv 0$, $\|f\|_E = 1$ and q_n is a polynomial of degree n for which $\|f-q_n\|_E \leqq 1+\varepsilon$, where $\varepsilon > 0$. Then, does $\|q_n\|_E$ approach zero as ε approaches zero and if so is there any relationship between their respective rates of convergence to zero. For example, is $\|q_n\|_E = O(\varepsilon^\beta)$ for some $\beta > 0$? We consider the real case first.

2. **The real case.** Our problem in the real case was settled in 1958 by G. Freud [**4**] who showed that $\|q_n\| = O(\varepsilon)$ where "O" depends only on E and f. His result also holds for approximation by generalized real valued polynomials (cf. Meinardus [**1**, p. 22]). We shall now state and prove Freud's result for ordinary polynomials and in the process describe

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in some way how "O" depends on E and f. Our proof shall also serve as a motivation for the corresponding proof in the complex case.

THEOREM 1. *Suppose f is continuous and real valued on E, a compact subset of the real line, $p_n(f, E)\equiv 0$, $\|f\|_E=1$ and q_n is a real polynomial of degree n for which $\|f-q_n\|_E<1+\varepsilon$ where $\varepsilon>0$. It then follows that $\|q_n\|_E=O(\varepsilon)$, where "O" depends only on E and f. Furthermore this estimate is sharp for each n.*

PROOF. By Chebyshev's Theorem [**1**, p. 20] there exist $n+2$ points $\{x_k\}_{k=1}^{n+2}$ in E such that

$$x_1 < x_2 < \cdots < x_{n+2}, \quad \text{and} \quad f(x_k) = -f(x_{k+1}) \qquad \text{for } k = 1, 2, \cdots, n+1.$$

We may assume without loss of generality that $f(x_k)=(-1)^k$, $k=1, 2, \cdots, n+2$. Let $w(x)=\prod_{k=1}^{n+2}(x-x_k)$, $M=\max_k |w'(x_k)|$ and $m=\min_k |w'(x_k)|$. We now claim that $|q_n(x_k)|<(n+1)M\varepsilon/m$, for $k=1, 2, \cdots, n+2$.

If for some j, $1\leqq j\leqq n+2$, $|q_n(x_j)|\geqq(n+1)M\varepsilon/n$, then for some $l\neq j$, $1\leqq l\leqq n+2$, $|q_n(x_l)|\geqq\varepsilon$ and

$$\text{sign}\left[\frac{q_n(x_l)}{w'(x_l)}\right] = -\text{sign}\left[\frac{q_n(x_j)}{w'(x_j)}\right]. \tag{1}$$

This follows, by Lagrange's interpolation formula since,

$$q_n(x) = w(x)\sum_{k=1}^{n+2}\frac{q_n(x_k)}{w'(x_k)(x-x_k)} = \left(\sum_{k=1}^{n+2}\frac{q_n(x_k)}{w'(x_k)}\right)x^{n+1} + \cdots,$$

and so $\sum_{k=1}^{n+2} q_n(x_k)/w'(x_k)=0$, since q_n is a polynomial of degree n. Now by the hypothesis of our theorem, $|f(x_k)-q_n(x_k)|\leqq 1+\varepsilon$ for $k=1, 2, \cdots, n+2$ and so

$$\begin{aligned} &\text{sign}[q_n(x_j)] = \text{sign}[f(x_j)] = \text{sign}[(-1)^j], \quad \text{and similarly,} \\ &\text{sign}[q_n(x_l)] = \text{sign}[f(x_l)] = \text{sign}[(-1)^l]. \end{aligned} \tag{2}$$

If we let, $l=t+j$, then $(-1)^l=(-1)^t(-1)^j$ and so by (2),

$$\text{sign}[q_n(x_l)] = \text{sign}[(-1)^t q_n(x_j)]. \tag{3}$$

However, sign$[w'(x_k)]$ alternates on E and, in particular,

$$\text{sign}[w'(x_l)] = \text{sign}[(-1)^t w'(x_j)]. \tag{4}$$

Thus by (3) and (4) we get

$$\operatorname{sign}\left[\frac{q_n(x_l)}{w'(x_l)}\right] = \operatorname{sign}\left[\frac{q_n(x_j)}{w'(x_j)}\right],$$

thus contradicting (1). Hence our claim follows.

Now since E is compact, the functions $\{w(x)/(x-x_k)\}_{k=1}^{n+2}$ are uniformly bounded on E, say by L, and so again using Lagrange's interpolation formula we can write

$$\begin{aligned}|q_n(x)| &= \left|\sum_{k=1}^{n+2} \frac{q_n(x_k)w(x)}{w'(x_k)(x-x_k)}\right| \leqq \sum_{k=1}^{n+2}\left|\frac{q_n(x_k)w(x)}{w'(x_k)(x-x_k)}\right| \\ &\leqq \frac{(n+2)(n+1)ML\varepsilon}{m^2}, \quad \text{for each } x \in E,\end{aligned}$$

where M and m are as before. Hence our theorem follows.

In order to demonstrate that the estimate $\|q_n\|_E = O(\varepsilon)$ is sharp for each n let $0 \leqq x_1 < x_2 < \cdots < x_{n+2} < 1$ and $E = \{x_k\}_{k=1}^{n+2} \cup \{1\}$. Define the function f on E by setting $f(x_k) = (-1)^k$, $k = 1, \cdots, n+2$, and $f(1) = 0$, and let $q_{n,\varepsilon}(x) = \varepsilon 2^n(x-\frac{1}{2})^n$. Then f, E and $q_{n,\varepsilon}$ satisfy the conditions of Theorem 1; however, $\|q_{n,\varepsilon}\|_E = |q_{n,\varepsilon}(1)| = \varepsilon$.

3. **The complex case.**

THEOREM 2. *Suppose f is continuous on E, a compact set in the plane, $n \in Z^+$, $p_n(f, E) \equiv 0$, $\|f\|_E = 1$ and q_n is a polynomial of degree n for which $\|f-q_n\|_E \leqq 1+\varepsilon$ where $1 > \varepsilon > 0$. It then follows that $\|q_n\|_E = O(\varepsilon^\beta)$, for every $\beta < \frac{1}{2}$, where "O" depends only on E, f and β. Furthermore this estimate is sharp for each n in that it is not in general true for $\beta = \frac{1}{2}$.*

PROOF. By the Remez condition [3, p. 437] there exists m distinct points $\{z_k\}_{k=1}^m$ in E and m positive constants $\{\lambda_k\}_{k=1}^m$, $2n+3 \geqq m \geqq n+2$, such that

$$\text{(i)}\ |f(z_k)| = 1 \quad \text{for } k = 1, 2, \cdots, m, \qquad \text{and}$$

$$\text{(ii)}\ \sum_{k=1}^m \lambda_k \overline{f(z_k)} z_k^j = 0, \quad \text{for } j = 0, 1, \cdots, n.$$

In particular,

$$\sum_{k=1}^m \lambda_k \overline{f(z_k)} q_n(z_k) = 0. \tag{5}$$

Set $\mu_k = f(z_k)$ for $k = 1, 2, \cdots, m$ and write $q_n(z_k) = (1+\alpha_k)\mu_k + i\beta_k\mu_k$, $k = 1, 2, \cdots, m$, where the α_k's and β_k's are real. This can be done in a unique manner. By the hypothesis of our theorem we have that

$$|\alpha_k\mu_k + i\beta_k\mu_k| \leqq 1 + \varepsilon, \quad \text{for } k = 1, 2, \cdots, m,$$

so as a consequence

$$\alpha_k^2 + \beta_k^2 < (1+\varepsilon)^2, \quad \text{and in particular,} \tag{6}$$
$$|\alpha_k| < (1+\varepsilon).$$

With this notation, the expression (5) can be rewritten

$$\sum_{k=1}^{m} \lambda_k \bar{\mu}_k[(1+\alpha_k)\mu_k + i\beta_k\mu_k] = 0,$$

or

$$\sum_{k=1}^{m} \lambda_k + \sum_{k=1}^{m} \lambda_k\alpha_k + i\sum_{k=1}^{m} \lambda_k\beta_k = 0.$$

Equating real parts yields

$$\sum_{k=1}^{m} \lambda_k\alpha_k = -\sum_{k=1}^{m} \lambda_k. \tag{7}$$

We now claim that for any $\beta<\frac{1}{2}$, $|q_n(z_k)|<\varepsilon^\beta$, for $k=1, 2, \cdots, m$, and all polynomials q_n satisfying the conditions of our theorem, if ε is sufficiently small. If for "sufficiently small" ε and some j, $1\leqq j\leqq m$, $|q_n(z_j)|>\varepsilon^\beta$, it will then follow that

$$1 + \alpha_j > (\varepsilon^{2\beta} - 2\varepsilon - \varepsilon^2)/2. \tag{8}$$

In order to demonstrate this we note that since $|q_n(z_j)|>\varepsilon^\beta$, we then have that $(1+\alpha_j)^2+\beta_j^2>\varepsilon^{2\beta}$, and from (6) we have $(1+\varepsilon)^2>\alpha_j^2+\beta_j^2$. Combining these two inequalities yields (8).

Now by (7) we have $\sum_{k=1;k\neq j}^{m} \lambda_k\alpha_k = -\sum_{k=1}^{m} \lambda_k - \alpha_j\lambda_j$, and so

$$\left|\sum_{k=1;k\neq j}^{m} \lambda_k\alpha_k\right| \geqq \sum_{k=1}^{m} \lambda_k + \alpha_j\lambda_j.$$

Recalling (6) that $|\alpha_k|<(1+\varepsilon)$ for $k=1, 2, \cdots, m$, we obtain

$$(1+\varepsilon)\sum_{k=1;k\neq j}^{m} \lambda_k > \sum_{k=1}^{m} \lambda_k + \alpha_j\lambda_j,$$

and so

$$\varepsilon \sum_{k=1;k\neq j}^{m} \lambda_k > (1+\alpha_j)\lambda_j > \lambda_j(\varepsilon^{2\beta} - 2\varepsilon - \varepsilon^2)/2. \tag{9}$$

This is impossible if ε is sufficiently small since $1-2\beta<0$; hence our claim follows if we note that expression (9) does not depend on q_n. As in Theorem 1, we can complete our proof and show that $\|q_n\|_E=O(\varepsilon^\beta)$ by applying the Lagrange interpolation formula.

In order to demonstrate the sharpness of our result we construct for each $n\in Z^+$ and each $M>0$ a set E and a function f which satisfy the conditions of our theorem and then construct for every sufficiently small ε a polynomial $q_{n,\varepsilon}(z)$ of degree n for which $\|f-q_{n,\varepsilon}\|_E \leqq 1+\varepsilon$ and such that $\|q_{n,\varepsilon}\|_E \geqq M\varepsilon^{1/2}$.

We choose $n+2$ points $\{z_k\}_{k=1}^{n+2}$ such that if $w(z)=\prod_{k=1}^{n+2}(z-z_k)$,

$$\left|\sum_{k=2}^{n+2}\frac{1}{w'(z_k)}\right| < \sum_{k=2}^{n+2}\frac{1}{|w'(z_k)|}, \quad \text{and let}$$

(10)
$$\tau_n(z) = w(z)\sum_{k=2}^{n+2}\frac{\mu w'(z_k) - |w'(z_k)|}{|w'(z_k)|\,w'(z_k)(z-z_k)},$$

where $\mu=(\sum_{k=2}^{n+2} 1/w'(z_k))/(\sum_{k=2}^{n+2} 1/|w'(z_k)|)$. The polynomial τ_n is not identically constant and so there exists z_0, $z_0\neq z_k$, $k=1, 2, \cdots, n+2$, such that $|\tau_n(z_0)|>M+1$. Let $E=\{z_k\}_{k=0}^{n+2}$ and define a function f on E by setting $f(z_0)=0$ and $f(z_k)=w'(z_k)/|w'(z_k)|$ for $k=1, 2, \cdots, n+2$. It then follows [2] that $p_n(f, E)\equiv 0$ and $\|f\|_E=1$.

Now for sufficiently small ε, let $\alpha=a(w'(z_1)/|w'(z_1)|)+ib(w'(z_1)/|w'(z_1)|)$, where $a=-(\varepsilon+\varepsilon^2)/2$ and $b=(\varepsilon-a^2)^{1/2}$. We define $q_{n,\varepsilon}$ by setting $q_{n,\varepsilon}(z)=(z-z_1)q_{n-1}(z)+\alpha$, where q_{n-1} is the polynomial of degree $n-1$ of best uniform approximation to the function $(f(z)-\alpha)/(z-z_1)$ on the set $\{z_k\}_{k=2}^{n+2}$ with respect to the weight function $|z-z_1|$. That is, q_{n-1} minimizes

$$\max_{2\leqq k\leqq n+2} |z_k - z_1|\,|(f(z_k)-\alpha)/(z_k-z_1) - p_{n-1}(z_k)|$$

for all polynomials p_{n-1} of degree $n-1$. Set

$$\delta_n = \max_{2\leqq k\leqq n+2} |z_k - z_1|\,|(f(z_k)-\alpha)/(z_k-z_1) - q_{n-1}(z_k)|.$$

Note that $\delta_n=\max_{2\leqq k\leqq n+2}|f(z_k)-q_{n,\varepsilon}(z_k)|$, and so let us first show that $\delta_n\leqq 1+\varepsilon$. By applying the work [2] of Motzkin and Walsh, δ_n can be calculated explicitly, in fact

$$\begin{aligned}\delta_n &= \left|\left(\sum_{k=2}^{n+2}\frac{f(z_k)-\alpha}{w'(z_k)}\right)\Big/\left(\sum_{k=2}^{n+2}\frac{1}{|w'(z_k)|}\right)\right| \\ &= \left|1-\alpha\left(\sum_{k=2}^{n+2}\frac{1}{w'(z_k)}\right)\Big/\left(\sum_{k=2}^{n+2}\frac{1}{|w'(z_k)|}\right)\right|.\end{aligned}$$

Now noting that $\sum_{k=1}^{n+2} 1/w'(z_k)=0$ and by our choice of the z_k's we can write

$$-\alpha\left(\sum_{k=2}^{n+2}\frac{1}{w'(z_k)}\right)\Big/\left(\sum_{k=2}^{n+2}\frac{1}{|w'(z_k)|}\right) = \alpha\sigma\left(\frac{|w'(z_1)|}{w'(z_1)}\right) = \sigma a + i\sigma b,$$

where $1>\sigma\geqq 0$. Thus

(11) $$\delta_n^2 \leqq (1+\sigma a)^2 + (\sigma b)^2 < (1+|a|)^2 + b^2 = (1+\varepsilon)^2.$$

By our choice of the value $q_{n,\varepsilon}(z_1)$, a straightforward calculation yields

$$|f(z_1) - q_{n,\varepsilon}(z_1)| = 1 + \varepsilon. \tag{12}$$

Again by appealing to [2] we can calculate

$$q_{n-1}(z) = \frac{w(z)}{(z - z_1)}\left[\sum_{k=2}^{n+2} \frac{f(z_k) - \alpha}{w'(z_k)(z - z_k)} - A_0 \sum_{k=2}^{n+2} \frac{1}{|w'(z_k)|\,(z - z_k)}\right],$$

where

$$A_0 = \left(\sum_{k=2}^{n+2} \frac{f(z_k) - \alpha}{w'(z_k)}\right) \Big/ \left(\sum_{k=2}^{n+2} \frac{1}{|w'(z_k)|}\right).$$

Now by substituting the given values for $f(z_k)$ we obtain $(z-z_1)q_{n-1}(z)= \alpha\tau_n(z)$. Hence,

$$\begin{aligned} \|q_{n,\varepsilon}\|_E &\geqq |q_n(z_0)| = |(z_0 - z_1)q_{n-1}(z_0) + \alpha| \\ &\geqq |\alpha\tau_n(z_0)| - |\alpha| \geqq (M + 1)\,|\alpha| - |\alpha| = M\varepsilon^{1/2}. \end{aligned}$$

Also, $|f(z_0)-q_{n,\varepsilon}(z_0)|=|q_{n,\varepsilon}(z_0)|\leqq 1+\varepsilon$ if ε is sufficiently small, and so by (11) and (12) we get that $\|f-q_{n,\varepsilon}\|_E\leqq 1+\varepsilon$ and our example is complete.

4. **Remark.** As a consequence of Theorem 2, if a function f is continuous on a compact set E then for each $\beta<\frac{1}{2}$ and $n\in Z^+$, there exists a least constant, M_n, such that if q_n is a polynomial of degree n for which $\|f-q_n\|_E\leqq\|f-p_n(f, E)\|_E\,(1+\varepsilon)$, where $0<\varepsilon<1$, then

$$\|p_n(f, E) - q_n\|_E \leqq \|f - p_n(f, E)\|_E\, M_n\varepsilon^{\beta}.$$

Whether the sequence $\{M_n\}_{n=0}^{\infty}$ is bounded for each f and E remains open. A similar question can be posed in the real case.

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