ON THE CONTINUITY OF BEST POLYNOMIAL APPROXIMATIONS

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Abstract. Suppose \(f\) is a continuous complex valued function defined on a compact set \(E\) in the plane and \(p_n(f, E)\) is the polynomial of degree \(n\) of best uniform approximation to \(f\) on \(E\). If a polynomial \(q_n\) of degree \(n\) approximates \(f\) on \(E\) "almost" as well as \(p_n(f, E)\), then \(q_n\) is "almost" \(p_n(f, E)\). Sharp estimates, one for the real and one for the general case, are found for \(\|q_n - p_n(f, E)\|_E\) in terms of the quantity \((\|f - q_n\|_E - \|f - p_n(f, E)\|_E)\), where \(\|\cdot\|_E\) denotes the uniform norm on \(E\).

1. Introduction. For a function \(f\) continuous on \(E\), a compact set in the plane, let \(\|f\|_E = \max_{z \in E} |f(z)|\). Also, for \(n \in \mathbb{Z}^+\), let \(p_n(f, E)\) denote the polynomial of degree \(n\) of best uniform approximation to \(f\) on \(E\). A basic question that arises in the theory of best approximation is:

If two continuous functions \(f_1\) and \(f_2\) are "close" on \(E\), are their polynomials of best approximation \(p_n(f_1, E)\) and \(p_n(f_2, E)\) also "close" on \(E\)?

More precisely, if \(\{f_m\}_{m=1}^{\infty}\) is a sequence of continuous functions converging uniformly to \(f\) on \(E\), does the sequence \(\{p_n(f_m, E)\}_{m=1}^{\infty}\) converge uniformly to \(p_n(f, E)\) on \(E\) (for each \(n\)) and if so, how rapid is the convergence.

The above problem can be stated in even greater generality. Suppose \(f\) is continuous on a compact set \(E\), \(n \in \mathbb{Z}^+\), \(p_n(f, E) \equiv 0\), \(\|f\|_E = 1\) and \(q_n\) is a polynomial of degree \(n\) for which \(\|f - q_n\|_E \leq 1 + \varepsilon\), where \(\varepsilon > 0\). Then, does \(\|q_n\|_E\) approach zero as \(\varepsilon\) approaches zero and if so is there any relationship between their respective rates of convergence to zero. For example, is \(\|q_n\|_E = O(\varepsilon^\beta)\) for some \(\beta > 0\)? We consider the real case first.

2. The real case. Our problem in the real case was settled in 1958 by G. Freud [4] who showed that \(\|q_n\| = O(\varepsilon)\) where "\(O\)" depends only on \(E\) and \(f\). His result also holds for approximation by generalized real valued polynomials (cf. Meinardus [1, p. 22]). We shall now state and prove Freud's result for ordinary polynomials and in the process describe

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in some way how "O" depends on $E$ and $f$. Our proof shall also serve as a motivation for the corresponding proof in the complex case.

**Theorem 1.** Suppose $f$ is continuous and real valued on $E$, a compact subset of the real line, $\|f\|_E \equiv 0$, $\|f\|_E = 1$ and $q_n$ is a real polynomial of degree $n$ for which $\|f - q_n\|_E < 1 + \varepsilon$ where $\varepsilon > 0$. It then follows that $\|q_n\|_E = O(\varepsilon)$, where "$O$" depends only on $E$ and $f$. Furthermore this estimate is sharp for each $n$.

**Proof.** By Chebyshev's Theorem [1, p. 20] there exist $n+2$ points $\{x_k\}_{k=1}^{n+2}$ in $E$ such that

$$x_1 < x_2 < \cdots < x_{n+2}, \quad \text{and} \quad f(x_k) = -f(x_{k+1})$$

for $k = 1, 2, \ldots, n+1$.

We may assume without loss of generality that $f(x_k) = (-1)^k$, $k = 1, 2, \ldots, n+2$. Let $w(x) = \prod_{k=1}^{n+2} (x - x_k)$, $M = \max_k |w'(x_k)|$ and $m = \min_k |w'(x_k)|$. We now claim that $|q_n(x_k)| < (n+1)M/e$ for $k = 1, 2, \ldots, n+2$.

If for some $j$, $1 \leq j \leq n+2$, $|q_n(x_j)| \geq (n+1)M/e$, then for some $l \neq j$, $1 \leq l \leq n+2$, $|q_n(x_l)| \geq e$ and

$$\operatorname{sign}[q_n(x_j)] = \operatorname{sign}[q_n(x_l)] = \operatorname{sign}[(n+1)M/e].$$

This follows, by Lagrange's interpolation formula since,

$$q_n(x) = \frac{w(x)}{\prod_{k=1}^{n+2} (x - x_k)} \sum_{k=1}^{n+2} \frac{q_n(x_k)}{w'(x_k)} (x - x_k) = \left( \sum_{k=1}^{n+2} \frac{q_n(x_k)}{w'(x_k)} x_{n+2} \right) + \cdots,$$

and so $\sum_{k=1}^{n+2} \frac{q_n(x_k)}{w'(x_k)} x_{n+2} = 0$, since $q_n$ is a polynomial of degree $n$. Now by the hypothesis of our theorem, $|f(x_k) - q_n(x_k)| \leq 1 + \varepsilon$ for $k = 1, 2, \ldots, n+2$ and so

$$\operatorname{sign}[q_n(x_j)] = \operatorname{sign}[f(x_j)] = \operatorname{sign}[(-1)^j], \quad \text{and similarly,}$$

$$\operatorname{sign}[q_n(x_j)] = \operatorname{sign}[f(x_j)] = \operatorname{sign}[(-1)^j].$$

If we let, $l = t+j$, then $(-1)^l = (-1)^t(-1)^j$ and so by (2),

$$\operatorname{sign}[q_n(x_j)] = \operatorname{sign}[(n+1)M/e].$$

However, $\operatorname{sign}[w'(x_k)]$ alternates on $E$ and, in particular,

$$\operatorname{sign}[w'(x_j)] = \operatorname{sign}[(n+1)w'(x_j)].$$
Thus by (3) and (4) we get
\[
\text{sign} \left[ \frac{q_n(x_j)}{w'(x_j)} \right] = \text{sign} \left[ \frac{q_n(x_j)}{w'(x_j)} \right],
\]
thus contradicting (1). Hence our claim follows.

Now since \( E \) is compact, the functions \( \{w(x)/(x-x_k)\}_{k=1}^{n+2} \) are uniformly bounded on \( E \), say by \( L \), and so again using Lagrange's interpolation formula we can write
\[
|q_n(x)| = \left| \sum_{k=1}^{n+2} \frac{q_n(x_k)w(x)}{w'(x_k)(x-x_k)} \right| \leq \sum_{k=1}^{n+2} \frac{|q_n(x_k)w(x)|}{w'(x_k)|x-x_k|} = (n+2)(n+1)MLε, \quad \text{for each } x \in E,
\]
where \( M \) and \( m \) are as before. Hence our theorem follows.

In order to demonstrate that the estimate \( \|q_n\|_E = O(ε) \) is sharp for each \( n \) let \( 0 ≤ x_1 < x_2 < \cdots < x_{n+2} < 1 \) and \( E = \{x_k\}_{k=1}^{n+2} \). Define the function \( f \) on \( E \) by setting \( f(x_k) = (-1)^k \), \( k = 1, \cdots, n+2 \), and \( f(1) = 0 \), and let \( q_{n,ε}(x) = ε2^n(x-\frac{1}{2})^n \). Then \( f \), \( E \) and \( q_{n,ε} \) satisfy the conditions of Theorem 1; however, \( \|q_{n,ε}\|_E = |q_{n,ε}(1)| = ε \).

3. The complex case.

**Theorem 2.** Suppose \( f \) is continuous on \( E \), a compact set in the plane, \( n \in \mathbb{Z}^+ \), \( p_n(f, E) ≡ 0 \), \( \|f\|_E = 1 \) and \( q_n \) is a polynomial of degree \( n \) for which \( \|f-q_n\|_E ≤ 1 + ε \) where \( 1 > ε > 0 \). It then follows that \( \|q_n\|_E = O(ε^β) \), for every \( β < \frac{1}{2} \), where "O" depends only on \( E \), \( f \) and \( β \). Furthermore this estimate is sharp for each \( n \) in that it is not in general true for \( β = \frac{1}{2} \).

**Proof.** By the Remez condition [3, p. 437] there exists \( m \) distinct points \( \{z_k\}_{k=1}^m \) in \( E \) and \( m \) positive constants \( \{λ_k\}_{k=1}^m \) with \( 2n+3 ≥ m ≥ n+2 \), such that
(i) \( |f(z_k)| = 1 \) for \( k = 1, 2, \cdots, m \), and
(ii) \( \sum_{k=1}^m λ_k f(z_k)z_k^j = 0 \). for \( j = 0, 1, \cdots, n \).
In particular,
\[
\sum_{k=1}^m λ_k f(z_k)q_n(z_k) = 0.
\]
Set \( μ_k = f(z_k) \) for \( k = 1, 2, \cdots, m \) and write \( q_n(z_k) = (1 + α_k)μ_k + iβ_kμ_k \), \( k = 1, 2, \cdots, m \), where the \( α_k \)'s and \( β_k \)'s are real. This can be done in a unique manner. By the hypothesis of our theorem we have that
\[
|α_kμ_k + iβ_kμ_k| ≤ 1 + ε, \quad \text{for } k = 1, 2, \cdots, m,
\]
so as a consequence

\[ \alpha_k^2 + \beta_k^2 < (1 + \varepsilon)^2, \]  
and in particular,

\[ |\alpha_k| < (1 + \varepsilon). \]  

With this notation, the expression (5) can be rewritten

\[ \sum_{k=1}^{m} \lambda_k \alpha_k [(1 + \alpha_k) \mu_k + i \beta_k \mu_k] = 0, \]

or

\[ \sum_{k=1}^{m} \lambda_k + \sum_{k=1}^{m} \lambda_k \alpha_k + i \sum_{k=1}^{m} \lambda_k \beta_k = 0. \]

Equating real parts yields

\[ \sum_{k=1}^{m} \lambda_k \alpha_k = - \sum_{k=1}^{m} \lambda_k. \]

We now claim that for any \( \beta < \frac{1}{2}, \) \( |q_n(z_j)| < \varepsilon^\theta, \) for \( k = 1, 2, \ldots, m, \) and all polynomials \( q_n \) satisfying the conditions of our theorem, if \( \varepsilon \) is sufficiently small. If for "sufficiently small" \( \varepsilon \) and some \( j, 1 \leq j \leq m, |q_n(z_j)| > \varepsilon^\theta, \) it will then follow that

\[ 1 + \alpha_j > (e^{2\theta} - 2\varepsilon - \varepsilon^\theta)/2. \]

In order to demonstrate this we note that since \( |q_n(z_j)| < \varepsilon^\theta, \) we then have that \( (1 + \alpha_j)^2 + \beta_j^2 > \varepsilon^\theta, \) and from (6) we have \( (1 + \varepsilon)^2 > \alpha_j^2 + \beta_j^2. \) Combining these two inequalities yields (8).

Now by (7) we have \( \sum_{k=1}^{m} \lambda_k \alpha_k = - \sum_{k=1}^{m} \lambda_k - \alpha_j \lambda_j, \) and so

\[ \left| \sum_{k=1}^{m} \lambda_k \alpha_k \right| \geq \sum_{k=1}^{m} \lambda_k + \alpha_j \lambda_j. \]

Recalling (6) that \( |\alpha_k| < (1 + \varepsilon) \) for \( k = 1, 2, \ldots, m, \) we obtain

\[ (1 + \varepsilon) \sum_{k=1}^{m} \lambda_k > \sum_{k=1}^{m} \lambda_k + \alpha_j \lambda_j, \]

and so

\[ \varepsilon \sum_{k=1}^{m} \lambda_k > (1 + \alpha_j) \lambda_j > \lambda_j (e^{2\theta} - 2\varepsilon - \varepsilon^\theta)/2. \]

This is impossible if \( \varepsilon \) is sufficiently small since \( 1 - 2\beta < 0; \) hence our claim follows if we note that expression (9) does not depend on \( q_n. \) As in Theorem 1, we can complete our proof and show that \( \|q_n\|_F = O(\varepsilon^\theta) \) by applying the Lagrange interpolation formula.

In order to demonstrate the sharpness of our result we construct for each \( n \in \mathbb{Z}^+ \) and each \( M > 0 \) a set \( E \) and a function \( f \) which satisfy the
conditions of our theorem and then construct for every sufficiently small 
\( \epsilon \) a polynomial \( q_{n,e}(z) \) of degree \( n \) for which \( \|f - q_{n,e}\|_{E} \leq 1 + \epsilon \) and such that \( \|q_{n,e}\|_{E} \geq M \epsilon^{1/2} \).

We choose \( n+2 \) points \( \{z_k\}_{k=1}^{n+2} \) such that if \( w(z) = \prod_{k=1}^{n+2} (z - z_k) \),

\[
\left| \sum_{k=2}^{n+2} \frac{1}{w'(z_k)} \right| < \sum_{k=2}^{n+2} \left| \frac{1}{w'(z_k)} \right|,
\]

and let

\[
\tau_n(z) = w(z) \sum_{k=2}^{n+2} \frac{\mu w'(z_k) - |w'(z_k)|}{w'(z_k)} w'(z_k)(z - z_k),
\]

where \( \mu = (\sum_{k=2}^{n+2} 1/w'(z_k))/(\sum_{k=2}^{n+2} 1/|w'(z_k)|) \). The polynomial \( \tau_n \) is not identically constant and so there exists \( z_0, z_0 \neq z_k, k = 1, 2, \ldots, n+2, \) such that \( |\tau_n(z_0)| > M+1 \). Let \( E = \{z_k\}_{k=0}^{n+2} \) and define a function \( f \) on \( E \) by setting \( f(z_0) = 0 \) and \( f(z_k) = w'(z_k)/|w'(z_k)| \) for \( k = 1, 2, \ldots, n+2 \). It then follows [2] that \( p_{n}(f, E) = 0 \) and \( \|f\|_{E} = 1 \).

Now for sufficiently small \( \epsilon \), let \( \alpha = a(w'(z_1)/|w'(z_1)|) + ib(w'(z_1)/|w'(z_1)|) \), where \( a = - (\epsilon + \epsilon^2)/2 \) and \( b = (\epsilon - \epsilon^2)^{1/2} \). We define \( q_{n,e} \) by setting \( q_{n,e}(z) = (z - z_1)q_{n-1}(z) + \alpha \), where \( q_{n-1} \) is the polynomial of degree \( n-1 \) of best uniform approximation to the function \( (f(z) - \alpha)/(z - z_1) \) on the set \( \{z_k\}_{k=2}^{n+2} \) with respect to the weight function \( |z - z_1| \). That is, \( q_{n-1} \) minimizes

\[
\max_{2 \leq k \leq n+2} |z_k - z_1| |(f(z_k) - \alpha)/(z_k - z_1) - p_{n-1}(z_k)|
\]

for all polynomials \( p_{n-1} \) of degree \( n-1 \). Set

\[
\delta_n = \max_{2 \leq k \leq n+2} |z_k - z_1| |(f(z_k) - \alpha)/(z_k - z_1) - q_{n-1}(z_k)|.
\]

Note that \( \delta_n = \max_{2 \leq k \leq n+2} |f(z_k) - q_{n,e}(z_k)| \), and so let us first show that \( \delta_n \leq 1 + \epsilon \). By applying the work [2] of Motzkin and Walsh, \( \delta_n \) can be calculated explicitly, in fact

\[
\delta_n = \left| \left( \sum_{k=2}^{n+2} f(z_k) - \alpha \right) \left/ \left( \sum_{k=2}^{n+2} w'(z_k) \right) \right| \right.
\]

\[
\left. = \left| 1 - \alpha \left( \sum_{k=2}^{n+2} \frac{1}{w'(z_k)} \right) \left/ \left( \sum_{k=2}^{n+2} \frac{1}{w'(z_k)} \right) \right| \right|.
\]

Now noting that \( \sum_{k=1}^{n+2} 1/w'(z_k) = 0 \) and by our choice of the \( z_k \)’s we can write

\[
-\alpha \left( \sum_{k=2}^{n+2} \frac{1}{w'(z_k)} \right) \left/ \left( \sum_{k=2}^{n+2} \frac{1}{w'(z_k)} \right) \right| = \alpha \frac{|w'(z_1)|}{w'(z_1)} = \sigma a + i \sigma b,
\]

where \( 1 > \sigma \geq 0 \). Thus

\[
\delta_n^2 \leq (1 + \sigma a^2 + (\sigma b)^2 < (1 + |a|)^2 + b^2 = (1 + \epsilon)^2.
\]
By our choice of the value \( q_{n,\varepsilon}(z_1) \), a straightforward calculation yields

\[ |f(z_1) - q_{n,\varepsilon}(z_1)| = 1 + \varepsilon. \tag{12} \]

Again by appealing to \([2]\) we can calculate

\[
q_{n-1}(z) = \frac{w(z)}{(z - z_1)} \left[ \sum_{k=2}^{n+2} \frac{f(z_k) - \alpha}{w'(z_k)(z - z_k)} - A_0 \sum_{k=2}^{n+2} \frac{1}{|w'(z_k)| (z - z_k)} \right],
\]

where

\[
A_0 = \left( \sum_{k=2}^{n+2} \frac{f(z_k) - \alpha}{w'(z_k)} \right) \left( \sum_{k=2}^{n+2} \frac{1}{|w'(z_k)|} \right).
\]

Now by substituting the given values for \( f(z_k) \) we obtain \((z - z_1)q_{n-1}(z) = \alpha \tau_n(z)\). Hence,

\[ \|q_{n,\varepsilon}\|_E \geq |q_n(z_0)| = |(z_0 - z_1)q_{n-1}(z_0) + \alpha| \geq |\alpha \tau_n(z_0)| - |\alpha| \geq (M + 1) |\alpha| - |\alpha| = M \varepsilon^{1/2}. \]

Also, \(|f(z_0) - q_{n,\varepsilon}(z_0)| = |q_{n,\varepsilon}(z_0)| \leq 1 + \varepsilon \) if \( \varepsilon \) is sufficiently small, and so by (11) and (12) we get that \( \| f - q_{n,\varepsilon} \|_E \leq 1 + \varepsilon \) and our example is complete.

4. Remark. As a consequence of Theorem 2, if a function \( f \) is continuous on a compact set \( E \) then for each \( \beta < \frac{1}{2} \) and \( n \in \mathbb{Z}^+ \), there exists a least constant, \( M_n \), such that if \( q_n \) is a polynomial of degree \( n \) for which

\[ \| f - q_n \|_E \leq \| f - p_n(f, E) \|_E (1 + \varepsilon), \]

where \( 0 < \varepsilon < 1 \), then

\[ \| p_n(f, E) - q_n \|_E \leq \| f - p_n(f, E) \|_E M_n \varepsilon^\beta. \]

Whether the sequence \( \{M_n\}_{n=0}^{\infty} \) is bounded for each \( f \) and \( E \) remains open. A similar question can be posed in the real case.

REFERENCES


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