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TYCHONOFF'S THEOREM FOR HYPERSPACES

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ABSTRACT. If $\exp(X_i)\backslash\{\varnothing\}$ is equipped with a topology that preserves the topological convergence of nets of sets for every $i\in I$, then the Tychonoff product of the family $\{\exp(X_i)\backslash\{\varnothing\}:i\in I\}$ is compact if and only if X_i is compact for every $i\in I$. A similar result concerning sequential compactness is valid, for countable I.

In [**8**], Y.-F. Lin describes a topology for the cartesian product, P, of the family $\{\exp(X_i)\backslash\{\varnothing\}:i\in I\}$ which he calls the M-product of the family $\{X_i:i\in I\}$ of topological spaces. He proves that the M-product is compact, if X_i is compact, for every $i\in I$. We improve this result (see Theorem 4) and obtain a related result (see Theorem 8) by observing that the M-product is a Tychonoff product, where $\exp(X_i)\backslash\{\varnothing\}$ is equipped with the Vietoris topology (which preserves the topological convergence of nets of sets, since X_i is compact), for every $i\in I$.

If X is a topological space with topology T, T_V denotes the Vietoris or finite topology on $\exp(X)$ generated by T. We note (see E. Michael [**9**]) that T_V can be generated by using the class

$$\{\{A\in\exp(X):A\subset U\}:U\in T\}\cup\{\{A\in\exp(X):A\cap V\neq\varnothing\}:V\in T\}$$

as a subbase. For convenience let $\exp^*(X)=\exp(X)\backslash\{\varnothing\}$.

The topology for the M-product has as a subbase the totality of all sets of either the form (a) $\{F\in P:F(i)\subset U_i\}$ or the form (b) $\{F\in P:F(j)\cap V_j\neq\varnothing\}$, where U_i and V_j are arbitrary open subsets of X_i and X_j, respectively. The images of (a) and (b) under their respective projection maps p_i and p_j are subbasic open sets in the Vietoris topology on the factor spaces $\exp^*(X_i)$ and $\exp^*(X_j)$, respectively. It is an easy exercise to show that the Tychonoff topology on the cartesian product of a family of spaces can be generated by the class of inverse images under the projection maps onto the factor spaces of the subbasic open sets in the factor spaces; see for instance [**2**].

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The following definition of the topological convergence of sequences of sets can be found in F. Hausdorff [6] and of nets of sets appeared for the first time in G. Birkhoff [1].

The topological limit inferior (respectively, superior) of a net or sequence (A_n, N) of subsets of a topological space X is that subset of X, denoted by $\mathrm{Li}(A_n)$ (respectively, $\mathrm{Ls}(A_n)$), each of whose elements satisfies the condition that for each of its neighborhoods U the set $\{n \in N : A_n \cap U \neq \varnothing\}$ is a residual (respectively, cofinal) subset of N. Recall that a subset C of a directed set D is cofinal (respectively, residual) in D means that for every $d \in D$ there is a $c \in C$ satisfying $c > d$ in D (respectively, means that $D \backslash C$ is not cofinal in D). We remark that if N is the positive integers equipped with their natural ordering, then cofinal translates as infinite and residual as cofinite (a cofinite subset meaning one whose complement is finite). The topological limit of (A_n, N), denoted by $\mathrm{Lim}(A_n)$, is said to exist if and only if $\mathrm{Li}(A_n) = \mathrm{Ls}(A_n)$. Whenever $\mathrm{Lim}(A_n)$ exists, (A_n, N) is said to converge topologically. The power set of X equipped with this convergence structure for nets of sets we will denote by the pair $(\exp(X), C)$. The pair $(\exp(X), C^*)$ will denote the power set of X equipped with the sequential convergence structure. To say that a subset $\bar{R}$ of $\exp(X)$ is C (C^*)-compact means every net (sequence) in $\bar{R}$ has a subnet (subsequence) which C (C^*)-converges to an element of $\bar{R}$. If T is a topology for $\exp(X)$ and $\bar{R}$ is a subset of $\exp(X)$, then by T on $\bar{R}$ is meant the relativization of T to $\bar{R}$.

(1) $\exp^*(X)$ *is C-compact if and only if X is compact.*

PROOF. Suppose $\exp^*(X)$ is C-compact and (x_n, N) is a net in X, then $(\{x_n\}, N)$ has a subnet $(\{x_{k(m)}\}, M)$ which C-converges to $A \in \exp^*(X)$. If $x \in A$, then $(x_{k(m)}, M)$ is a subnet of (x_n, N) convergent in X to x, as $x \in \mathrm{Li}(\{x_{k(m)}\})$ (see Z. Frolik [5]). Conversely, suppose X is compact and (A_n, N) is a net in $\exp^*(X)$. Now (A_n, N) has a subnet $(A_{k(m)}, M)$ which is C-convergent in $\exp(X)$, since S. Mrowka (see [10]) has shown that $\exp(X)$ is C-compact for every topological space X. Let $x_{k(m)} \in A_{k(m)}$, for every $m \in M$, then $(x_{k(m)}, M)$ has a cluster point x in X. Thus $\mathrm{Lim}(A_{k(m)}) = \mathrm{Ls}(A_{k(m)}) \neq \varnothing$, since $x \in \mathrm{Ls}(A_{k(m)})$ (see [5]).

To say that a topology T for $\exp(X)$ preserves C (C^*) for $\exp(X)$ means that a net (sequence) (A_n, N) C (C^*)-converges to A implies (A_n, N) T-converges to A.

As a corollary to (1) we have

(2) $\exp^*(X)$ *is compact in a topology that preserves C if and only if X is compact.*

PROOF. Suppose $\exp^*(X)$ is compact in a topology T on $\exp^*(X)$, where T preserves C. Assuming that X is not compact implies there is a net (A_n, N) in $\exp^*(X)$ which C-converges to $\varnothing$; thus every subnet of (A_n, N) also C-converges to $\varnothing$. Hence every subnet of (A_n, N) T-converges to $\varnothing$, as T preserves C. This contradiction yields that X is compact. Conversely, if X is compact, then $\exp^*(X)$ is C-compact, hence is compact in any topology that preserves C.

We now consider a particular class of topologies for $\exp(X)$ that will preserve C (C^*). To say that a topology for $\exp(X)$ is of finite type means it has a base of sets of the form $[K; \bar{M}]=\{A \in \exp(X): A\cap K=\varnothing$ and $A\cap V\neq\varnothing$, for every $V\in\bar{M}\}$, where $\bar{M}$ is always a finite collection of nonempty open subsets of X and $K\in\bar{S}\subset\exp(X)$ and $\bar{S}$ is closed under finite unions and contains $\varnothing$. Such a collection is always a base since $[K_1; \bar{M}_1]\cap[K_2; \bar{M}_2]=[K_1\cup K_2; \bar{M}_1\cup\bar{M}_2]$ and $\exp(X)=[\varnothing, \{X\}]$. The concept of a topology of finite type is a subclass of the class of $\underline{R}$-$\underline{S}$ topologies defined by Mrowka in **[10]**.

We remark that if $\bar{S}$ equals, respectively, the class of closed sets, the class of compact sets, and the class of compact closures of open sets, then the topology generated is the Vietoris topology, the H-topology (see Fell **[4]**), and the lbc-topology (see Mrowka **[10]**). We denote these topologies by the labels T_V, T_F and T_M, respectively.

We note that $\{\varnothing\}=[\varnothing, \varnothing]$, so $\varnothing$ is always an isolated point in a finite type topology for $\exp(X)$, thus in T_V, T_F and T_M. We note further that $T_M\subset T_F$ always; $T_F\subset T_V$ for Hausdorff spaces, since then compact implies closed; $T_V\subset T_F$ for compact spaces, since then closed implies compact; and $T_F\subset T_M$ on the subspace of $\exp(X)$ consisting of the closed subsets of X (which we denote by $[\exp(X)]$) for locally compact regular spaces X that need not be Hausdorff. To be consistent $[\exp^*(X)]$ will denote the collection of nonempty closed subsets of X.

(3) *If T is a topology for* $\exp(X)$ *of finite type and $\bar{S}$ is a subclass of the class of compact subsets of X, then T preserves C.*

PROOF. Suppose the net (A_n, N) in $\exp(X)$ C-converges to A and $A\in[K, \bar{M}]$, where $\bar{M}=\{V_1, \cdots, V_t\}$. Then $A\cap K=\varnothing$ and $A\cap V_i\neq\varnothing$, for $i=1, 2, \cdots, t$. For each $i=1, 2, \cdots, t$, choose $x_i\in V_i\cap A$, then $x_i\in \mathrm{Li}(A_n)=A$, $x_i\in V_i$, V_i is open; so there is an $m_i\in N$ satisfying $A_n\cap V_i\neq\varnothing$, for every $n\geqq m_i$. Choose $m'\in N$ so that $m'\geqq m_i$ for every $i=1, 2, \cdots, t$, then $A_n\cap V_i\neq\varnothing$, for every $n\geqq m'$ and for every $i=1, 2, \cdots, t$. Suppose for every $n\in N$ there is an $m\in N$ satisfying $m>n$ and $A_m\cap K\neq\varnothing$. Let (B_m, M) be the subnet of (A_n, N) so defined, i.e., $B_m\cap K\neq\varnothing$, for every $m\in M$. For every $m\in M$ choose $y_m\in B_m\cap K$, then (y_m, M) is a subnet of K, hence has a convergent subnet (z_e, E), since K

is compact. Suppose (z_e, E) converges to $z \in K$, then $z \in \mathrm{Ls}(B_m, M) \subset \mathrm{Ls}(A_n, N) = A$, see Frolik [5]. Thus the contradiction that $z \in A \cap K$ implies there is an $m^* \in N$ satisfying $A_n \cap K = \varnothing$, for every $n \geqq m^*$. So (A_n, N) T-converges to A, as $A_n \in [K, \bar{M}]$ for every $n \geqq m$, where m is chosen so that $m \in N$, $m \geqq m'$ and $m \geqq m^*$.

Thus T_M and T_F preserve C always and T_V preserves C, if X is compact. The value of (3) stems from the fact that if X is compact and not regular, then T_F and T_V are different topologies for $\exp(X)$, even if X is T_1 (i.e., singleton sets are closed). This follows since Fell [4] has shown that T_F is Hausdorff on $[\exp(X)]$ if X is locally compact, and since Michael [9] has shown that T_V is Hausdorff on $[\exp^*(X)]$ if and only if X is regular.

We remark that (2) and (3) above imply that if X is compact, then $\exp^*(X)$ is compact in each of the topologies T_V, T_F and T_M, since $\exp^*(X)$ is closed in every topology of finite type.

It might be interesting to study the properties of topologies of finite type for $\exp(X)$, but this will not be done in this paper.

The Tychonoff theorem now yields.

(4) THEOREM. *If T_i is a topology on* $\exp^*(X_i)$ *that preserves C_i, for every $i \in I$, then the Tychonoff product of* $\{\exp^*(X_i) : i \in I\}$ *is compact if and only if X_i is compact for every $i \in I$.*

We remark that Lin's theorem is the reverse implication in (4), where $\exp^*(X_i)$ is equipped with the Vietoris topology for every $i \in I$, since, by the remark following (3), $\exp^*(X)$ is compact in the Vietoris topology if and only if X is compact. For a deeper examination of this last equivalence, see [7].

To obtain some related results on sequences of sets and countable collections of spaces we must restrict our attention to the class of hypersequentially compact (HSC) spaces, i.e., those spaces for which $\exp(X)$ is C^*-compact. We remark that the Sorgenfrey plane is not HSC and that every hereditary Lindelof space is HSC. For more details on HSC spaces see [3]. We note that a sequentially compact space is one in which every sequence has a convergent subsequence and that in what follows N always denotes $\{1, 2, 3, \cdots\}$ and (A_n) is always a sequence of sets.

(5) $\exp^*(X)$ *is C^*-compact if and only if X is sequentially compact and HSC.*

PROOF. Suppose $\exp^*(X)$ is C^*-compact and (x_n) is a sequence in X. Now $(\{x_n\})$ has a subsequence $(\{x_{k(n)}\})$ which C^*-converges to $A \in \exp^*(X)$. If $x \in A$, then $(x_{k(n)})$ is a subsequence of (x_n) convergent in X to x, as $x \in \mathrm{Li}(\{x_{k(n)}\})$. To see that X is HSC, let (A_n) be any sequence in $\exp(X)$ and $M = \{n \in N : A_n = \varnothing\}$. If M is infinite, then $(A_m)_{m \in M}$ is a subsequence of (A_n) that converges to $\varnothing$. If M is finite, then $(A_n)_{n \geq t}$ is a subsequence of (A_n) belonging to $\exp^*(X)$; hence it and consequently (A_n) has a C^*-convergent subsequence, where $t \in N$ and $t > m$, for every $m \in M$.

Conversely, suppose X is sequentially compact and HSC and (A_n) is a sequence in $\exp^*(X)$. Now (A_n) has a subsequence $(A_{k(n)})$ which is C^*-convergent in $\exp(X)$. Let $x_{k(n)} \in A_{k(n)}$, for every $n \in N$, then $(x_{k(n)})$ has a *sequential cluster point* (i.e., limit point of a subsequence), x, in X. Thus $\operatorname{Lim}(A_{k(n)}) = \operatorname{Ls}(A_{k(n)}) \neq \varnothing$, as $x \in \operatorname{Ls}(A_{k(n)})$.

As a corollary to (2) we have

(6) $\exp^*(X)$ *is sequentially compact in a topology that preserves* C^* *if and only if* X *is sequentially compact and HSC.*

We remark that a proof for (6) can be obtained by replacing, in the proof for (2), net and C by sequence and C^*, respectively, and compact by either sequentially compact and HSC or sequentially compact, where appropriate. (7) below simply shows that there is a topology for $\exp(X)$ that preserves C^*.

(7) *If* T *is a topology for* $\exp(X)$ *of finite type and* $\bar{S}$ *is a subclass of the class of sequentially compact subsets of* X, *then* T *preserves* C^*.

PROOF. Parrot the proof of (3) replacing net by sequence, subset by subsequence and compact by sequentially compact, whenever necessary.

An application of the countable diagonal process yields

(8) THEOREM. *If* T_n *is a topology on* $\exp^*(X_n)$ *that preserves* C_n^*, *for every* $n \in N$, *then the Tychonoff product of* $\{\exp^*(X_n) : n \in N\}$ *is sequentially compact if and only if* X_n *is sequentially compact and HSC, for every* $n \in N$.

(9) COROLLARY. *If* X_n *is sequentially compact and hereditarily Lindelof, then the Tychonoff product of* $\{\exp^*(X_n) : n \in N\}$ *is sequentially compact.*

We remark that if X is a T_1 space, (1) through (9) remain valid with $\exp^*(X)$ replaced by $[\exp^*(X)]$, since $\operatorname{Lim}(A_n) = \operatorname{cl}(\operatorname{Lim}(A_n)) = \operatorname{Lim}(\operatorname{cl}(A_n))$.

REFERENCES

1. G. Birkhoff, *Moore-Smith convergence in general topology*, Ann. of Math. (2) **38** (1937), 39–56.

2. D. Bushaw, *Elements of general topology*, Wiley, New York, 1963, Theorem 14, p. 52. MR **28** #2515.

3. F. A. Chimenti, *On the sequential compactness of the space of subsets*, Bull. Polon. Acad. Sci. **20** (1972).

4. J. M. G. Fell, *A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space*, Proc. Amer. Math. Soc. **13** (1962), 472–476. MR **25** #2573.

5. Z. Frolik, *Concerning topological convergence of sets*, Czechoslovak Math. J. **10 (85)** (1960), 168–180. MR **22** #7098.

6. F. Hausdorff, *Mengenlehre*, 3rd ed., de Gruyter, Berlin, 1937; English transl., Chelsea, New York, 1962, p. 168. MR **25** #4999.

7. James Keesling, *On the equivalence of normality and compactness in hyperspaces*, Pacific J. Math. **33** (1970), 657–667. MR **42** #2418.

8. Y.-F. Lin, *Tychonoff's theorem for the space of multifunctions*, Amer. Math. Monthly **74** (1967), 399–400. MR **35** #969.

9. E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182. MR **13**, 54.

10. S. Mrowka, *Some comments on the space of subsets*, Lecture Notes in Math., vol. 171, Springer-Verlag, Berlin and New York, 1970, pp. 59–63. MR **42** #5216.

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