PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 1, January 1973

ON THE BORDISM RING OF COMPLEX PROJECTIVE SPACE

CLAUDE SCHOCHET

ABSTRACT. The bordism ring $MU_*(CP^\infty)$ is central to the theory of formal groups as applied by D. Quillen, J. F. Adams, and others recently to complex cobordism. In the present paper, rings $E_*(CP^\infty)$ are considered, where E is an oriented ring spectrum, $R=\pi_*(E)$, and $pR=0$ for a prime p. It is known that $E_*(CP^\infty)$ is freely generated as an R-module by elements $\{\beta_r|r\geqq 0\}$. The ring structure, however, is not known. It is shown that the elements $\{\beta_{p^r}|r\geqq 0\}$ form a simple system of generators for $E_*(CP^\infty)$ and that $\beta_{p^r}^p\equiv s^{p^r}\beta_{p^r} \bmod(\beta_1,\cdots,\beta_{p^r-1})$ for an element $s\in R$ (which corresponds to $[CP^{p-1}]$ when $E=MUZ_p$). This may lead to information concerning $E_*(K(Z,n))$.

1. **Introduction.** Let E be an associative, commutative ring spectrum with unit, with $R=\pi_*E$. Then E determines a generalized homology theory E_* and a generalized cohomology theory E^* (as in G. W. Whitehead [5]). Following J. F. Adams [1] (and using his notation throughout), assume that E is *oriented* in the following sense:

There is given an element $x\in \tilde{E}^*(CP^\infty)$ such that $\tilde{E}^*(S^2)$ is a free R-module on $i^*(x)$, where $i:S^2=CP^1\to CP^\infty$ is the inclusion.

(The Thom-Milnor spectrum MU which yields complex bordism theory and cobordism theory satisfies these hypotheses and is of seminal interest.) By a spectral sequence argument and general nonsense, Adams shows:

(1.1) $E^*(CP^\infty)$ is the graded ring of formal power series $R[[x]]$.

(1.2) The map $m:E^*(CP^\infty)\to E^*(CP^\infty\times CP^\infty)$ induced by the group multiplication on $CP^\infty=K(Z,2)$ gives $E^*(CP^\infty)$ and $E_*(CP^\infty)$ the structure of commutative, cocommutative Hopf algebras over R.

Writing the coproduct $m:R[[x]]\to R[[x_1,x_2]]$ by $m(x)=\mu(x_1,x_2)=\sum a_{ij}x_1^ix_2^j$, (1.2) implies

(1.3) m is R-linear and satisfies the equations

$$\mu(x_1,0)=x_1,\qquad \mu(0,x_2)=x_2,$$
$$\mu(x_1,\mu(x_2,x_3))=\mu(\mu(x_1,x_2),x_3),$$
$$\mu(x_1,x_2)=\mu(x_2,x_1).$$

Received by the editors April 14, 1972.

AMS (MOS) subject classifications (1970). Primary 57D90, 57F05; Secondary 14L05.

Key words and phrases. Complex bordism, complex cobordism, oriented spectrum, graded formal group, Hopf algebra over a ring.

Condition (1.3) is precisely the statement that m is a *formal product* and that $(E^*(CP^\infty), m)$ is a *formal group*. Recent work by Quillen [3] and others has indicated the great strength of formal group techniques in studying complex bordism and cobordism. It therefore seems reasonable to attain to a very firm grasp on the Hopf algebra $E_*(CP^\infty)$, which is central to bordism applications.

Let $\langle\ ,\ \rangle : E^*(CP^\infty)\otimes E_*(CP^\infty)\to R$ be the Kronecker pairing. There are unique elements $\beta_n \in E_*(CP^\infty)$ such that $\langle x^i, \beta_n\rangle = \delta^i_n$. Adams proves:

(1.4) $E_*(CP^\infty)$ is a free R-module on generators $1=\beta_0, \beta_1, \cdots, \beta_n, \cdots$.

(1.5) The coproduct Ψ on $E_*(CP^\infty)$ is determined by $\Psi(\beta_n)=\sum_{i=0}^n \beta_i\otimes\beta_{n-i}$.

The remaining open problem is the expression of the *algebra* structure of $E_*(CP^\infty)$ in some reasonable way. This would be of some conceptual interest, and it would also be of practical interest, for example, in the computation of $E_*(K(Z,3))$ via the Eilenberg-Moore spectral sequence with $E_2=\mathrm{Tor}_{E_*(CP^\infty)}(R,R)$.

Our approach to the problem is via restriction to

2. **Hopf algebras of prime characteristic.** Henceforth assume that $pR=0$ for a fixed prime p. (For example, one could take $\boldsymbol{E}$ to be the spectrum corresponding to the theory $MU_*(\ ;Z_p)$, where Z_p is the field of p elements.)

(2.1) Definition. An augmented R-algebra A is said to have the ordered set $y_1, y_2, \cdots, y_n, \cdots$ as a *simple system of generators* if the monomials

$$\{y_{j_1}^{t_1}\cdots y_{j_k}^{t_k} \mid j_1<j_2<\cdots<j_k \text{ and } 0<t_i<p\}$$

form a free R-basis for A, and if for each n, only finitely many y_j have degree n.

Let QA denote the R-module of indecomposable elements of A; i.e. $QA=IA/(IA)^2$ where IA is the kernel of the augmentation $A\to R$.

(2.2) Main Theorem. (a) *The elements* $\beta_1, \beta_p, \cdots, \beta_{p^n}, \cdots$ *form a free R-basis for* $QE_*(CP^\infty)$.

(b) $\beta_{p^n}^p \equiv s^{p^n}\beta_{p^n} \bmod(\beta_1, \cdots, \beta_{p^{n-1}})$ *where* $s=\sum_{i=1}^{p-1} a_{i,1}\langle x^i, \beta_1^{p-1}\rangle \in R$.

(c) *If* $\deg(x)<0$, *then the elements* $\beta_1, \beta_p, \cdots, \beta_{p^n}, \cdots$ *form a simple system of generators for* $E_*(CP^\infty)$.

The proof of the Main Theorem is simple and purely algebraic, resting upon a decomposition theorem for Hopf algebras of the type $E_*(CP^\infty)$ which is stated below and proved in [4]. We now present the algebraic setting for

3. **The Decomposition Theorem.** Let R be a graded commutative ring with $R_n=0$ if n is negative, $pR=0$ for a prime integer p, and x an indeterminate of nonpositive degree. Define F to be the free R-module on generators $1=\beta_0, \beta_1, \cdots, \beta_n, \cdots$ with $\deg(\beta_k)=-\deg(x^k)$. Give F the structure of an R-coalgebra via the Whitney coproduct

$$\Psi(\beta_n) = \sum_{i=0}^{n} \beta_i \otimes \beta_{n-i} \tag{3.1}$$

and define the Kronecker pairing $\langle\ ,\ \rangle: R[[x]]\otimes F \to R$ by $\langle x^i, \beta_n\rangle=\delta^i_n$. Let $m: R[[x]]\to R[[x_1, x_2]]$ be a cocommutative formal product (satisfying (1.3)), so that F becomes a commutative, cocommutative Hopf algebra over R.

Let F_0 be the R-subalgebra of F generated by β_1. Direct calculation shows

$$\beta_1^p = s\beta_1 \quad \text{in } F \tag{3.2}$$

where $s=\sum_{i=1}^{p-1} a_{i,1}\langle x^i, \beta_1^{p-1}\rangle \in R$. Hence F_0 is isomorphic as an R-algebra to the polynomial algebra generated over R by $\bar{\beta}_1$ modulo the relation $\bar{\beta}_1^p=s\bar{\beta}_1$. In fact, the isomorphism is as Hopf algebras, if we assume $\bar{\beta}_1$ to be primitive.

Define F_n to be the polynomial algebra generated over R by $\bar{\beta}_{p^n}$ (where $\deg(\bar{\beta}_k)=\deg(\beta_k)$) modulo the relation

$$\bar{\beta}_{p^n} = s^{p^n}\bar{\beta}_{p^n} \tag{3.3}$$

with the Hopf algebra (over R) structure obtained by declaring $\bar{\beta}_{p^n}$ to be primitive.

(3.4) DECOMPOSITION THEOREM. *There exists a diagram of Hopf algebras and morphisms of Hopf algebras over R:*

$$\begin{array}{ccc} F_0 \to F = & G_0 \\ & \downarrow \\ F_1 \quad \to & G_1 \\ & \downarrow \\ F_2 \quad \to & G_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array} \tag{3.5}$$

such that for each $n\geqq 1$,

$\mathscr{A}_n$: $G_n=G_{n-1}//F_{n-1}=G_{n-1}\otimes_{F_{n-1}}R$ *is the free R-module on generators $1=\beta_0, \beta_{p^n}, \beta_{2p^n}, \cdots, \beta_{kp^n}, \cdots$ (which are the images of $\beta_{kp^n}\in G_0$ under projection),*

$\mathscr{B}_n$: *The map* $F_n \to G_n$ *is given by* $\bar{\beta}_{p^n} \to \beta_{p^n}$ *and is an inclusion of Hopf algebras,*

and consequently, $\lim_n G_n = R$ (*on the identity* β_0).

4. **Proof of the Main Theorem.** Setting $F = E_*(CP^\infty)$, it suffices to prove the following, purely algebraic

(4.1) THEOREM. *With the notation and assumptions of* §3:

(a) *The elements* $\beta_1, \beta_p, \cdots, \beta_{p^n}, \cdots$ *form a free* R-*basis for* QF.

(b) $\beta_{p^n}^p \equiv s^{p^n} \beta_{p^n} \bmod (\beta_1, \cdots, \beta_{p^{n-1}})$.

(c) *If* $\deg(x) < 0$, *then the elements* $\beta_1, \beta_p, \cdots, \beta_{p^n}, \cdots$ *form a simple system of generators for* F.

Part (a) is immediate from diagram (3.5) and the definition of F_n. For part (b), pass to $G_n = F/(\beta_1, \cdots, \beta_{p^{n-1}})$ and observe that $\beta_{p^n}^p = s^{p^n} \beta_{p^n}$ there. Part (c) requires induction upon $\deg(\beta_{j_1}^{t_1} \cdots \beta_{j_k}^{t_k})$. Note that if $\deg(x) = 0$, as in the case of complex K-theory, then the monomials $\beta_{j_1}^{t_1} \cdots \beta_{j_k}^{t_k}$ still provide a free R-basis for F, but the finiteness part of (2.1) is not satisfied.

5. **Acknowledgments.** Kamata [2] proves (2.2)(a) and part of (2.2)(c) for the case $E = MUZ_p$ and also has some results on $MU_*(CP^\infty)$. His techniques are quite different. The author is deeply grateful to Alex Zabrodsky for his generous advice and assistance.

REFERENCES

1. J. F. Adams, *Quillen's work on formal groups and complex cobordism*, Lectures, University of Chicago, Chicago, Ill., 1970.

2. M. Kamata, *On the ring structure of* $U_*(BU(1))$, Osaka J. Math. 7 (1970), 417–422. MR **43** #1204.

3. D. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298. MR **40** #6565.

4. C. Schochet, *On the structure of graded formal groups of finite characteristic*, Proc. Cambridge Philos. Soc. (to appear).

5. G. W. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. **102** (1962), 227–283. MR **25** #573.

DEPARTMENT OF MATHEMATICS, HEBREW UNIVERSITY, JERUSALEM, ISRAEL

Current address: Department of Mathematics, Indiana University, Bloomington, Indiana 47401