TRIANGULAR MATRIX ALGEBRAS
OVER HENSEL RINGS

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Abstract. Let $(R, m)$ be a local Hensel ring and $A$ an algebra over $R$ which is finitely generated and projective as an $R$-module. If $A$ contains a complete set of mutually orthogonal primitive idempotents $e_1, \cdots, e_n$ indexed so that $e_i Ne_j = (0)$ whenever $i \geq j$, we show that $A$ is isomorphic to a generalized triangular matrix algebra and that $A$ is the epimorphic image of a finitely generated, projective $R$-algebra $B$ of Hochschild dimension less than or equal to one.

Introduction. The class of residue algebras of semiprimary hereditary algebras has been thoroughly discussed in [5], [9], [2], [6] and [12]. They consist of those finitely generated algebras $A$ over a field $R$ which contain a complete set of mutually orthogonal primitive idempotents $e_1, \cdots, e_n$ which can be indexed so that $e_i Ne_j = (0)$ whenever $i \geq j$, where $N$ denotes the Jacobson radical of $A$, and $A/N$ is $R$-separable. All such algebras are isomorphic to generalized triangular matrix algebras.

The purpose of this paper is to show that every finitely generated, projective algebra $A$ over a local Hensel ring $(R, m)$ satisfies the “triangular” idempotent condition $e_i Ne_j \subseteq mA$ whenever $i \geq j$ if and only if $A$ is a residue algebra of a finitely generated, projective algebra $B$ of Hochschild dimension less than or equal to one. We will call such algebras “almost one-dimensional.”

Such an algebra $B$, the maximal algebra for $A$, is usually neither semiprimary nor hereditary. In fact, if we assume that $R$ is also noetherian, it has been shown in [11] that $B$ is hereditary if and only if $R$ is a field ($R$-dim $B \leq 1$) or a DVR ($R$-dim $B = 0$); $B$ is semiprimary if and only if $R$ has a nilpotent radical—in which case every finitely generated, projective algebra over $R$ has infinite global dimension—or $R$ is a field.

A generalization of a result of S. U. Chase [2] will provide a sort of converse to Theorem 2.7 of [10] in the case of a Hensel ring:
Theorem 0. Let $A$ be a finitely generated, projective algebra over a commutative ring $R$. If $R$-dim $A = 1$, then, for every ideal $I$ containing $mA$ or some maximal ideal $m$ of $R$, $R$-dim($A/I$) is finite.

Conventions. Throughout this paper, all rings will have one and all ring homomorphisms will take the identity to the identity. All modules are unitary. By a local ring $R$, we mean only that $R$ has a unique maximal ideal $m$. By a finitely generated $R$-algebra or a projective $R$-algebra, we shall mean that $A$ is finitely generated or projective as a module over $R$. All homological dimensions will be taken as left modules and $R$-dim $A = \text{hd}_A(A)$ will denote the Hochschild (or cohomological) dimension of the algebra $A$, where $A^e = A \otimes_R A^o$. $N$ will always denote the Jacobson radical of the algebra $A$.

1. Almost one-dimensional algebras. We shall say that a finitely generated, projective algebra $A$ over a local Hensel ring $(R, m)$ is almost one-dimensional if every complete set of mutually orthogonal primitive idempotents $e_1, \cdots, e_n$ can be indexed so that $e_i N e_j \subseteq mA$ whenever $i \geq j$.

The definition is justified by the following results.

Theorem 1. Let $A$ be a finitely generated, projective algebra over a local Hensel ring $R$. (a) If $R$-dim $A \leq 1$, then $A$ is almost one-dimensional. (b) If $A$ is almost one-dimensional, then $A/mA$ is triangular in the sense of Chase; i.e., $A/mA$ contains a complete set of mutually orthogonal primitive idempotents $e_1, \cdots, e_n$ indexed so that $e_i(N/mA)e_j = (0)$ whenever $i \geq j$. (c) If $A$ is almost one-dimensional, then $R$-dim $A$ is finite.

(We note that the hypotheses that $R$ is Hensel and $A$ is $R$-projective are not necessary to prove (a) or (b).)

Proof. (a) is contained in Theorems 3.4 and 3.6 of [10]. (b) is obvious. For (c), an application of (b) and Theorem 4.1 of [2] shows that $R/m$-dim $A/mA = \text{gl dim} A/mA$ is finite. But then, by 2.1 of [10], $R$-dim $A = R/m$-dim $A/mA$ is finite.

By means of part (b) of Theorem 1 and the idempotent lifting theorem of G. Azumaya [1, Theorem 24], it is easy to deduce the following equivalent conditions for an algebra to be almost one-dimensional from Theorem 4.1 of [2].

Theorem 2. If $A$ is a finitely generated, projective algebra over a local Hensel ring $R$, the following conditions are equivalent:

(a) $A$ is almost one-dimensional.

(b) There exists a complete set of mutually orthogonal primitive idempotents $e_1, \cdots, e_n$ of $A$ which can be indexed so that $e_i N e_j \subseteq mA$ whenever $i \geq j$. 
(c) For every ideal I of A containing mA, A/I has finite Hochschild dimension.

d) A/(N^n + mA) has finite Hochschild dimension.

Results of [2], [5] and [9] on the nilpotence degree of the Jacobson radical are translatable in terms of the nilpotence degree of N modulo mA by use of the same techniques.

2. Construction of the maximal algebra. If A is almost one-dimensional, we shall call the one-dimensional algebra B which we are going to construct the "maximal algebra" for A.

Recall that a generalized triangular matrix algebra $T_n(A_1; M_{ij}/R)$ is the algebra defined in the following way: the $A_i$ are algebras over R, the $M_{ij}$ are left $A_i$- and right $A_j$-bimodules with $M_{ij} = 0$ for $i > j$ and $M_{ii} = A_i$; $R$ commutes with the $M_{ij}$. Multiplication is defined via homomorphisms $\phi^i_{ij} : M_{ii} \otimes A_i \otimes M_{ij} \to M_{ij}$ with the $\phi^i_{ij}$ and the $\phi^j_{ij}$ isomorphisms; these mappings satisfy the "associative" law: $\phi^j_{ik}(id_{ij} \otimes \phi^i_{jk}) = \phi^i_{jk}(\phi^j_{ik} \otimes id_{ij})$. These functions induce a matrix multiplication on

$$T_n(A_1; M_{ij}/R) = \left\{ \begin{pmatrix} a_{11} & m_{12} & \cdots & m_{1n} \\
 a_{22} & \cdots & m_{2n} \\
 \vdots & \ddots & \ddots & \ddots \\
 0 & \cdots & a_{nn} \end{pmatrix} : a_{ii} \text{ in } A_i, m_{ij} \text{ in } M_{ij} \right\}$$

(cf. [10] or [6, p. 465]).

**Theorem 3.** Let $A$ be a finitely generated, projective algebra over a local Hensel ring R. If $A$ is almost one-dimensional, then $A$ is isomorphic to a generalized triangular matrix algebra over R.

**Proof.** The proof is essentially contained in the case where $A/N$ is isomorphic to a direct sum of division algebras over R. In this case, the $e_i$ are all nonisomorphic. Now $e_iAe_i$ is a separable, projective R-algebra. So $S = \sum_{i=1}^n e_iAe_i$ is the inertial subalgebra whose existence is guaranteed by [10, Proposition 2.5] and [1, Theorem 33], since $S/mS = A/N$.

Furthermore, if $j > k$, then

$$e_jAe_k = e_jSe_k + e_jNe_k \subseteq \mathfrak{m} \left( \bigoplus_{i \neq j} e_iAe_i \right),$$

where the first sum is not direct. Thus $e_jAe_k = \mathfrak{m}(e_jAe_k)$. Hence by Nakayama's lemma, $e_jAe_k = (0)$ for all $j > k$. Hence under the natural maps induced by the multiplication in $A$ from $e_iAe_i$ and $e_iAe_j$ to $e_iAe_j$, we have that $A \cong T_n(e_iAe_i; e_iAe_j/R)$.

The general case follows by noting that if $e = \sum_{i=1}^n e_{ii}$, where $i$ denotes the distinct isomorphism classes of primitive idempotents, then by
[6, Proposition 2]: $R/m$-dim $e(A/mA)e = gl$ dim $e(A/mA)e = gl$ dim $A/mA = R/m$-dim $A/mA$. Again, by Theorem 2.1 of [10], $R$-dim $A = R$-dim $eAe$. $eAe$ is almost one-dimensional and $eAe/(eNe)$ is isomorphic to a direct sum of division algebras. Finally, following the techniques of [10] and [6], we have that if $eAe \cong T_n(e_iAe_{ij}; e_iAe_{ij}/R)$, then $A \cong T_n(A_i; M_{ij}/R)$ where $A_i = (e_iAe_{ii})_{s_i \times s_i}$ and $M_{ij} = (e_iAe_{ij})_{s_i \times s_j}$, the $s_i$ by $s_j$ matrices with entries from $e_iAe_{ii}$.

We are now in a position to construct the maximal algebra for $A$. Let $N^* = \sum_{i < k} M_{jk}, P = \sum_{i=1}^{n-1} M_{i, i+1}$, and $M = \sum_{i+1 < k} M_{ik}$. Clearly, $A = S \oplus N^* = S \oplus P \oplus M$ as $S$-$S$ bimodules, where $P$, $M$, and $N^*$ are finitely generated, projective $R$-modules. Set $P^{(k)}$ equal to the $k$-fold tensor product of $P$ with itself over $S$. The middle-four-interchange gives that $(P \otimes S P) \otimes R/m \cong P/mP \otimes S/mS P/mP$. An easy induction then shows that $P^{(k)} \otimes P^{(k)} \cong (P/mP)^{(2k)}$.

Let $B = S \oplus P \oplus P^{(2)} \oplus P^{(3)} \oplus \cdots S \oplus P \oplus T$ be the algebra with multiplication defined as in a graded ring with $S = P^{(0)}$; i.e.,

$$(p_1 \otimes \cdots \otimes p_n) \cdot (p_1' \otimes \cdots \otimes p_n') = (p_1 \otimes \cdots \otimes p_n \otimes p_1' \otimes \cdots \otimes p_n')$$

(cf. [8, Definition 1.4]). To apply Theorem 2.1 of [10], we need to know that $T$ is finitely generated and projective as an $R$-module. That $T$ is $R$-projective follows by noting that $P$ is $R$-projective and hence by [3, Proposition 2.3] $S$-projective; whence $P \otimes S P$ is $S$-projective and therefore $R$-projective. That $T$ is finitely generated is shown by the following: Since $P/mP$ is the $S/mS$-complement of $(N/mA)^2$ in $(N/mA)$, by [9, p. 71], $(P/mP)^{(n)} = (0)$ for some $n$; so by Nakayama’s lemma, $(P/mP)^{(n)} = (0)$. Finite generation is now clear.

But $B/mB$ is the maximal algebra over the triangular algebra $A/mA$. Hence, $R$-dim $B \leq 1$. Defining $f: B \rightarrow A$ by $f(s, p, p_1 \otimes p_2, \cdots ) = s + p + p_1 p_2 + \cdots$, we obtain an algebra epimorphism of $B$ onto $A$. It is clear that $B/T$ is isomorphic to $A/N^*$. Thus we have just shown

**Theorem 4.** Let $A$ be a finitely generated, projective algebra over a local Hensel ring $R$. The following are equivalent.

(a) $A$ is almost one-dimensional.

(b) There is a finitely generated, projective algebra $B$ over $R$ such that $R$-dim $B \leq 1$, $A$ is an epimorphic image of $B$ with $B/T \cong A/N^*$, where $N^*$ and $T$ are the squares of the $R$-complements of the inertial subalgebras of $A$ and $B$ respectively.

Combining Theorems 2 and 4, one obtains the complete analogue of the results of [9], [2] and [6] for finitely generated triangular algebras over a field.
3. **Miscellaneous results and corollaries.** In [4, p. 311] S. Eilenberg gave necessary and sufficient conditions for a finitely generated algebra over a field to have a given Hochschild dimension. This characterization involved the Jacobson radical of the algebra. We note the following extension of that result: (Recall that an inertial subalgebra of an algebra $A$ is a separable subalgebra $S$ of $A$ such that $A = S + N$, where the sum is not necessarily direct.)

**Theorem 5.** Let $A$ be a finitely generated, projective algebra over a local ring $R$. Suppose that $A$ has finite Hochschild dimension. Let $S$ be an inertial subalgebra of $A$ such that $A = S \oplus I$ as an $R$-direct sum for some ideal $I$ of $A$. Then the following hold: (a) $R$-dim $A = \text{hd} A(S)$; and (b) $R$-dim $A = 1 + \text{hd} A(I)$.

**Proof.** Since $A$ and $S$ are $R$-projective, by the argument of Theorem 2.1 of [10], and by the corollary to Theorem 3 of [4], one sees that the following equalities hold:

\[
R\text{-dim } A = R/m\text{-dim } A/mA = \text{hd} A/mA(A/N) = \text{hd} A/mA(S/mS) = \text{hd} A(S).
\]

The second part follows directly from the first.

**Corollary 5.1.** Let the setting be as in the theorem. Then $A$ is one-dimensional if and only if $I$ is projective as an $A$-module; $A$ is $R$-separable if and only if $I = (0)$.

In particular, the setting of Theorem 5 always holds true for the algebras considered in this paper. It gives us a particularly interesting characterization of which almost one-dimensional algebras are actually one-dimensional:

**Corollary 5.2.** Suppose $A$ is a finitely generated, projective, almost one-dimensional algebra over a local Hensel ring $R$. Then, $A = S \oplus N^*$, where $N^*$ is an ideal of $A$, and $\text{hd} A(S) = R$-dim $A$. Moreover, $A$ has Hochschild dimension one if and only if $N^*$ is projective as an $A$-module.

Finally, one can give the following characterization of almost one-dimensional algebras based upon the ideal $N^*$ rather than the Jacobson radical, which allows one to restate Theorem 2 in terms of $N^*$.

**Theorem 6.** Let $A$ be a finitely generated, projective algebra over a local Hensel ring $R$; let $A$ have finite Hochschild dimension. $A$ is almost one-dimensional if and only if there is an ideal $N^*$ of $A$ such that $A = S \oplus N^*$ (direct sum as $R$-modules), where $S$ is the inertial subalgebra of $A$, and there exists a complete set of mutually orthogonal primitive idempotents $e_1, \ldots, e_n$ such that $e_iN^*e_i = (0)$ for $i \geq j$. 

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Proof. The "only if" part is a direct consequence of Theorem 3. Suppose that $A = S \oplus N^*$, and $A$ contains the required set of idempotents. Then $e_i Ne_j = e_i(mS + N^*)e_j = e_i(mS)e_j \leq mA$ whenever $i \geq j$ and $N$ denotes the Jacobson radical of $A$.

Concluding Remarks. The algebras of Hochschild dimension one over a local Hensel ring act as the semiprimary hereditary algebras over a field. In fact, an almost one-dimensional algebra $A$ and its maximal algebra $B$ can easily be seen to satisfy the definition of quasi-cyclic algebras and related algebras (with $N^*$ and $T$ replacing the Jacobson radicals) as given by Hochschild in [7, pp. 369 and 372].

It would be useful to know that the ideal $N^*$ was a radical of the algebra $A$. In a future paper, we will show that if $R$ is a noetherian domain which is a local Hensel ring, then $N^*$ is the (Baer) lower radical of $A$. This is not true if $R$ is a complete local ring with nilpotent radical, in which case the Jacobson and lower radicals are equal.

Bibliography


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