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TRIANGULAR MATRIX ALGEBRAS OVER HENSEL RINGS

JOSEPH A. WEHLEN

ABSTRACT. Let $(R, \boldsymbol{m})$ be a local Hensel ring and A an algebra over R which is finitely generated and projective as an R-module. If A contains a complete set of mutually orthogonal primitive idempotents $e_1, \cdots, e_n$ indexed so that $e_iNe_j \subseteq \boldsymbol{m}A$ whenever $i \geqq j$, we show that A is isomorphic to a generalized triangular matrix algebra and that A is the epimorphic image of a finitely generated, projective R-algebra B of Hochschild dimension less than or equal to one.

Introduction. The class of residue algebras of semiprimary hereditary algebras has been thoroughly discussed in [5], [9], [2], [6] and [12]. They consist of those finitely generated algebras A over a field R which contain a complete set of mutually orthogonal primitive idempotents $e_1, \cdots, e_n$ which can be indexed so that $e_iNe_j=(0)$ whenever $i \geqq j$, where N denotes the Jacobson radical of A, and A/N is R-separable. All such algebras are isomorphic to generalized triangular matrix algebras.

The purpose of this paper is to show that every finitely generated, projective algebra A over a local Hensel ring $(R, \boldsymbol{m})$ satisfies the "triangular" idempotent condition $e_iNe_j \subseteq \boldsymbol{m}A$ whenever $i \geqq j$ if and only if A is a residue algebra of a finitely generated, projective algebra B of Hochschild dimension less than or equal to one. We will call such algebras "almost one-dimensional."

Such an algebra B, the maximal algebra for A, is usually neither semiprimary nor hereditary. In fact, if we assume that R is also noetherian, it has been shown in [11] that B is hereditary if and only if R is a field (R-dim $B \leqq 1$) or a DVR (R-dim $B=0$); B is semiprimary if and only if R has a nilpotent radical—in which case every finitely generated, projective algebra over R has infinite global dimension—or R is a field.

A generalization of a result of S. U. Chase [2] will provide a sort of converse to Theorem 2.7 of [10] in the case of a Hensel ring:

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THEOREM 0. *Let A be a finitely generated, projective algebra over a commutative ring R. If R-dim $A=1$, then, for every ideal I containing $\boldsymbol{m}A$ or some maximal ideal $\boldsymbol{m}$ of R, R-dim(A/I) is finite.*

CONVENTIONS. Throughout this paper, all rings will have one and all ring homomorphisms will take the identity to the identity. All modules are unitary. By a local ring R, we mean only that R has a unique maximal ideal $\boldsymbol{m}$. By a finitely generated R-algebra or a projective R-algebra, we shall mean that A is finitely generated or projective as a module over R. All homological dimensions will be taken as left modules and R-dim $A=\mathrm{hd}_{A^e}(A)$ will denote the Hochschild (or cohomological) dimension of the algebra A, where $A^e=A\otimes_R A^{\mathrm{op}}$. N will always denote the Jacobson radical of the algebra A.

1. **Almost one-dimensional algebras.** We shall say that a finitely generated, projective algebra A over a local Hensel ring $(R, \boldsymbol{m})$ is *almost one-dimensional* if every complete set of mutually orthogonal primitive idempotents $e_1, \cdots, e_n$ can be indexed so that $e_iNe_j\subseteq \boldsymbol{m}A$ whenever $i\geqq j$. The definition is justified by the following results.

THEOREM 1. *Let A be a finitely generated, projective algebra over a local Hensel ring R.* (a) *If R-dim $A\leqq 1$, then A is almost one-dimensional.* (b) *If A is almost one-dimensional, then $A/\boldsymbol{m}A$ is triangular in the sense of Chase; i.e., $A/\boldsymbol{m}A$ contains a complete set of mutually orthogonal primitive idempotents $e_1, \cdots, e_n$ indexed so that $e_i(N/\boldsymbol{m}A)e_j=(0)$ whenever $i\geqq j$.* (c) *If A is almost one-dimensional, then R-dim A is finite.*

(We note that the hypotheses that R is Hensel and A is R-projective are not necessary to prove (a) or (b).)

PROOF. (a) is contained in Theorems 3.4 and 3.6 of [**10**]. (b) is obvious. For (c), an application of (b) and Theorem 4.1 of [**2**] shows that $R/\boldsymbol{m}$-dim $A/\boldsymbol{m}A=\mathrm{gl\,dim}\, A/\boldsymbol{m}A$ is finite. But then, by 2.1 of [**10**], R-dim $A=R/\boldsymbol{m}$-dim $A/\boldsymbol{m}A$ is finite.

By means of part (b) of Theorem 1 and the idempotent lifting theorem of G. Azumaya [**1**, Theorem 24], it is easy to deduce the following equivalent conditions for an algebra to be almost one-dimensional from Theorem 4.1 of [**2**].

THEOREM 2. *If A is a finitely generated, projective algebra over a local Hensel ring R, the following conditions are equivalent:*

(a) *A is almost one-dimensional.*

(b) *There exists a complete set of mutually orthogonal primitive idempotents $e_1, \cdots, e_n$ of A which can be indexed so that $e_iNe_j\subseteq \boldsymbol{m}A$ whenever $i\geqq j$.*

(c) *For every ideal I of A containing $\boldsymbol{m}A$, A/I has finite Hochschild dimension.*

(d) *$A/(N^2+\boldsymbol{m}A)$ has finite Hochschild dimension.*

Results of [**2**], [**5**] and [**9**] on the nilpotence degree of the Jacobson radical are translatable in terms of the nilpotence degree of N modulo $\boldsymbol{m}A$ by use of the same techniques.

2. **Construction of the maximal algebra.** If A is almost one-dimensional, we shall call the one-dimensional algebra B which we are going to construct the "maximal algebra" for A.

Recall that a generalized triangular matrix algebra $T_n(A_i; M_{ij}/R)$ is the algebra defined in the following way: the A_i are algebras over R, the M_{ij} are left A_i- and right A_j-bimodules with $M_{ij}=(0)$ for $i>j$ and $M_{ii}=A_i$; R commutes with the M_{ij}. Multiplication is defined via homomorphisms $\phi^t_{ij}: M_{it}\otimes_{A_t} M_{tj}\to M_{ij}$ with the ϕ^t_{it} and the ϕ^i_{it} isomorphisms; these mappings satisfy the "associative" law: $\phi^j_{ik}(\mathrm{id}_{ij}\otimes\phi^t_{jk})=\phi^t_{ik}(\phi^j_{it}\otimes\mathrm{id}_{tk})$. These functions induce a matrix multiplication on

$$T_n(A_i{:}M_{ij}/R) = \left\{\begin{pmatrix} a_{11} & m_{12} & \dots & m_{1n} \\ & a_{22} & \dots & m_{2n} \\ & & & \vdots \\ 0 & & & a_{nn}\end{pmatrix} : a_{ii} \text{ in } A_i,\ m_{ij} \text{ in } M_{ij}\right\}$$

(cf. [**10**] or [**6**, p. 465]).

THEOREM 3. *Let A be a finitely generated, projective algebra over a local Hensel ring R. If A is almost one-dimensional, then A is isomorphic to a generalized triangular matrix algebra over R.*

PROOF. The proof is essentially contained in the case where A/N is isomorphic to a direct sum of division algebras over R. In this case, the e_i are all nonisomorphic. Now e_iAe_i is a separable, projective R-algebra. So $S=\sum_{i=1}^n e_iAe_i$ is the inertial subalgebra whose existence is guaranteed by [**10**, Proposition 2.5] and [**1**, Theorem 33], since $S/\boldsymbol{m}S=A/N$.

Furthermore, if $j>k$, then

$$e_jAe_k = e_jSe_k + e_jNe_k \subseteq \boldsymbol{m}\left(\oplus \sum_{i,t} e_iAe_t\right),$$

where the first sum is not direct. Thus $e_jAe_k=\boldsymbol{m}(e_jAe_k)$. Hence by Nakayama's lemma, $e_jAe_k=(0)$ for all $j>k$. Hence under the natural maps induced by the multiplication in A from e_iAe_t and e_tAe_j to e_iAe_j, we have that $A\cong T_n(e_iAe_i; e_iAe_j/R)$.

The general case follows by noting that if $e=\sum_{i=1}^n e_{i1}$, where i denotes the distinct isomorphism classes of primitive idempotents, then by

[**6**, Proposition 2]: $R/\mathbf{m}$-dim $e(A/\mathbf{m}A)e = \text{gl dim } e(A/\mathbf{m}A)e = \text{gl dim } A/\mathbf{m}A = R/\mathbf{m}$-dim $A/\mathbf{m}A$. Again, by Theorem 2.1 of [**10**], R-dim $A = R$-dim eAe. eAe is almost one-dimensional and eAe/eNe is isomorphic to a direct sum of division algebras. Finally, following the techniques of [**10**] and [**6**], we have that if $eAe \cong T_n(e_{i1}Ae_{i1}; e_{i1}Ae_{j1}/R)$, then $A \cong T_n(A_i; M_{ij}/R)$ where $A_i = (e_{i1}Ae_{i1})_{s_i \times s_i}$ and $M_{ij} = (e_{i1}Ae_{j1})_{s_i \times s_j}$, the s_i by s_j matrices with entries from $e_{i1}Ae_{j1}$. $\square$

We are now in a position to construct the maximal algebra for A. Let $N^* = \sum_{j<k} M_{jk}$, $P = \sum_{i=1}^{n-1} M_{i,i+1}$, and $M = \sum_{j+1<k} M_{jk}$. Clearly, $A = S \oplus N^* = S \oplus P \oplus M$ as S-S bimodules, where P, M, and N^* are finitely generated, projective R-modules. Set $P^{(k)}$ equal to the k-fold tensor product of P with itself over S. The middle-four-interchange gives that $(P \otimes_S P) \otimes R/\mathbf{m} \cong P/\mathbf{m}P \otimes_{S/\mathbf{m}S} P/\mathbf{m}P$. An easy induction then shows that $P^{(k)}/\mathbf{m}P^{(k)} \cong (P/\mathbf{m}P)^{(k)}$.

Let $B = S \oplus P \oplus P^{(2)} \oplus P^{(3)} \oplus \cdots = S \oplus P \oplus T$ be the algebra with multiplication defined as in a graded ring with $S = P^{(0)}$; i.e.,

$$(p_1 \otimes \cdots \otimes p_q) \cdot (p_1' \otimes \cdots \otimes p_s') = (p_1 \otimes \cdots \otimes p_q \otimes p_1' \otimes \cdots \otimes p_s')$$

(cf. [**8**, Definition 1.4]). To apply Theorem 2.1 of [**10**], we need to know that T is finitely generated and projective as an R-module. That T is R-projective follows by noting that P is R-projective and hence by [**3**, Proposition 2.3] S-projective; whence $P \otimes_S P$ is S-projective and therefore R-projective. That T is finitely generated is shown by the following: Since $P/\mathbf{m}P$ is the $S/\mathbf{m}S$-complement of $(N/\mathbf{m}A)^2$ in $(N/\mathbf{m}A)$, by [**9**, p. 71], $(P/\mathbf{m}P)^{(n)} = (0)$ for some n; so by Nakayama's lemma, $P^{(n)} = (0)$. Finite generation is now clear.

But $B/\mathbf{m}B$ is the maximal algebra over the triangular algebra $A/\mathbf{m}A$. Hence, R-dim $B \leqq 1$. Defining $f: B \to A$ by $f(s, p, p_1 \otimes p_2, \cdots) = s + p + p_1 p_2 + \cdots$, we obtain an algebra epimorphism of B onto A. It is clear that B/T is isomorphic to A/N^{*2}. Thus we have just shown

THEOREM 4. *Let A be a finitely generated, projective algebra over a local Hensel ring R. The following are equivalent.*

(a) *A is almost one-dimensional.*

(b) *There is a finitely generated, projective algebra B over R such that R-dim $B \leqq 1$, A is an epimorphic image of B with $B/T \cong A/N^{*2}$, where N^{*2} and T are the squares of the R-complements of the inertial subalgebras of A and B respectively.*

Combining Theorems 2 and 4, one obtains the complete analogue of the results of [**9**], [**2**] and [**6**] for finitely generated triangular algebras over a field.

3. **Miscellaneous results and corollaries.** In [**4**, p. 311] S. Eilenberg gave necessary and sufficient conditions for a finitely generated algebra over a field to have a given Hochschild dimension. This characterization involved the Jacobson radical of the algebra. We note the following extension of that result: (Recall that an inertial subalgebra of an algebra A is a separable subalgebra S of A such that $A=S+N$, where the sum is not necessarily direct.)

THEOREM 5. *Let A be a finitely generated, projective algebra over a local ring R. Suppose that A has finite Hochschild dimension. Let S be an inertial subalgebra of A such that $A=S\oplus I$ as an R-direct sum for some ideal I of A. Then the following hold*: (a) $R\text{-dim}\, A=\text{hd}_A(S)$; *and* (b) $R\text{-dim}\, A= 1+\text{hd}_A(I)$.

PROOF. Since A and S are R-projective, by the argument of Theorem 2.1 of [**10**], and by the corollary to Theorem 3 of [**4**], one sees that the following equalities hold:

$$\begin{aligned} R\text{-dim}\, A &= R/\boldsymbol{m}\text{-dim}\, A/\boldsymbol{m}A = \text{hd}_{A/\boldsymbol{m}A}(A/N) \\ &= \text{hd}_{A/\boldsymbol{m}A}(S/\boldsymbol{m}S) = \text{hd}_A(S). \end{aligned}$$

The second part follows directly from the first.

COROLLARY 5.1. *Let the setting be as in the theorem. Then A is one-dimensional if and only if I is projective as an A-module; A is R-separable if and only if $I=(0)$.*

In particular, the setting of Theorem 5 always holds true for the algebras considered in this paper. It gives us a particularly interesting characterization of which almost one-dimensional algebras are actually one-dimensional:

COROLLARY 5.2. *Suppose A is a finitely generated, projective, almost one-dimensional algebra over a local Hensel ring R. Then, $A=S\oplus N^*$, where N^* is an ideal of A, and $\text{hd}_A(S)=R\text{-dim}\, A$. Moreover, A has Hochschild dimension one if and only if N^* is projective as an A-module.*

Finally, one can give the following characterization of almost one-dimensional algebras based upon the ideal N^* rather than the Jacobson radical, which allows one to restate Theorem 2 in terms of N^*.

THEOREM 6. *Let A be a finitely generated, projective algebra over a local Hensel ring R; let A have finite Hochschild dimension. A is almost one-dimensional if and only if there is an ideal N^* of A such that $A=S\oplus N^*$ (direct sum as R-modules), where S is the inertial subalgebra of A, and there exists a complete set of mutually orthogonal primitive idempotents $e_1, \cdots, e_n$ such that $e_iN^*e_j=(0)$ for $i\geqq j$.*

PROOF. The "only if" part is a direct consequence of Theorem 3. Suppose that $A=S\oplus N^*$, and A contains the required set of idempotents. Then $e_iNe_j=e_i(mS+N^*)e_j=e_i(mS)e_j\subseteq mA$ whenever $i\geqq j$ and N denotes the Jacobson radical of A.

CONCLUDING REMARKS. The algebras of Hochschild dimension one over a local Hensel ring act as the semiprimary hereditary algebras over a field. In fact, an almost one-dimensional algebra A and its maximal algebra B can easily be seen to satisfy the definition of quasi-cyclic algebras and related algebras (with N^* and T replacing the Jacobson radicals) as given by Hochschild in [7, pp. 369 and 372].

It would be useful to know that the ideal N^* was a radical of the algebra A. In a future paper, we will show that if R is a noetherian domain which is a local Hensel ring, then N^* is the (Baer) lower radical of A. This is not true if R is a complete local ring with nilpotent radical, in which case the Jacobson and lower radicals are equal.

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DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO 45701