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COMMUTING ANALYTIC FUNCTIONS WITHOUT FIXED POINTS

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ABSTRACT. Let A be the set of nonidentity analytic functions which map the open unit disk into itself. Wolff has shown that the iterates of $f \in A$ converge uniformly on compact sets to a constant $T(f)$, unless f is an elliptic conformal automorphism of the disk. This paper presents a proof that if f and g are in A and commute under composition, and if f is not a hyperbolic conformal automorphism of the disk, then $T(f)=T(g)$. This extends, in a sense, a result of Shields. The proof involves the so-called angular derivative of a function in A at a boundary point of the disk.

Let D be the open unit disk in the complex plane. Let $\bar{D}$ be its closure. Shields [5] has proved the following result.

THEOREM 1. *If f and g are continuous in $\bar{D}$, analytic in D, and map $\bar{D}$ into itself, and if $f \circ g = g \circ f$, then f and g have a common fixed point.*

Let A be the set of all analytic functions which map D into D, except for the identity function, which we exclude. This paper presents an extension of the result of Shields to the set A.

For $f \in A$ we define the iterates of f recursively by $f^1 = f$, and $f^{n+1} = f \circ f^n$ when $n \in Z^+$. A member of A which maps D univalently onto D will be called a conformal automorphism (c.a.) of D. We shall assume that the reader is acquainted with the standard classification of linear fractional transformations as elliptic, hyperbolic, parabolic, or loxodromic, as given in [3, p. 70]. Each c.a. of D is of one of the first three types mentioned. The elliptic transformations yield noneuclidean rotations of D with the hyperbolic metric, while the hyperbolic and parabolic transformations have their fixed points on the boundary of D.

THEOREM 2 (WOLFF [7]). *If $f \in A$ is not an elliptic c.a. of D, then there is a constant $T(f) \in \bar{D}$ for which $\lim_{n\to\infty} f^n = T(f)$ uniformly on compact sets.*

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If $T(f)$ is in D, then it is a fixed point of f; and if f has a fixed point in D, it has only one, and that fixed point is $T(f)$. The preceding sentence is still true if we extend the definition of T so that when f is an elliptic c.a. of D, then $T(f)$ is the fixed point of f in D.

Suppose that f and g are in A, that $T(f) \in D$, and that $f \circ g = g \circ f$. Then

$$g[T(f)] = g(f[T(f)]) = f(g[T(f)]),$$

so $g[T(f)]$ is a fixed point for f. This implies $T(f) = g[T(f)]$, and consequently $T(g) = T(f)$. We wish to extend this result, as far as possible, to functions f for which $T(f) \notin D$. If two linear fractional transformations have the same set of fixed points, then they commute [**3**, p. 72]. There exist pairs f, g of hyperbolic conformal automorphisms of D which have a common set of fixed points, but for which $T(f) \neq T(g)$. For example

$$f(z) = (z - \tfrac{1}{2})/(1 - \tfrac{1}{2}z) \quad \text{and} \quad g(z) = (z + \tfrac{1}{2})/(1 + \tfrac{1}{2}z)$$

defines such a pair. Thus it is not true in general that commuting members of A satisfy $T(f) = T(g)$, but we shall show that the only exceptional cases involve pairs of hyperbolic conformal automorphisms of D.

The only cases in which linear fractional transformations commute without having the same fixed points involve certain pairs of elliptic transformations which cannot occur as pairs in A. Thus two conformal automorphisms of D commute if and only if they have the same fixed points.

For ease of notation when $f \in A$ and $|\zeta| = 1$ we shall write $f(\zeta) = \eta$ to mean $\lim_{r \to 1^-} f(r\zeta) = \eta$, and in that case we write $f'(\zeta) = u$ to mean

$$\lim_{r \to 1^-} \frac{f(r\zeta) - \eta}{r\zeta - \zeta} = u.$$

We shall have several occasions to use Julia's lemma, which we state in a form which is essentially that of [**4**, p. 58].

Theorem 3. *If* $|\zeta| = |\eta| = 1$, *and* $f \in A$ *satisfies* $f(\zeta) = \eta$, *then there is an extended real number* c, *with* $0 < c \leqq \infty$, *such that*

$$\lim_{z \to \zeta} \left| \frac{f(z) - \eta}{z - \zeta} \right| = c$$

as z *approaches* ζ *through any open triangle in* D *with vertex at* ζ. *If* c *is finite then* $f'(\zeta) = c\eta/\zeta$.

If c is finite, then it is also true that $\lim_{z \to \zeta} f'(z) = f'(\zeta)$ when z approaches ζ as in Theorem 3.

If $f \in A$ and $|\zeta|=1$, then $T(f)=\zeta$ if and only if $f(\zeta)=\zeta$ and $f'(\zeta)\leqq 1$. (See [6] for example.)

We shall use the following result of Lindelöf [2, p. 19].

THEOREM 4. *Let f be analytic and bounded in D. If $f(z)$ approaches w as z approaches $e^{i\theta}$ along some curve γ lying in D except for its terminal point at $e^{i\theta}$, then $f(z)$ approaches w uniformly as z approaches $e^{i\theta}$ in any open triangle in D with vertex $e^{i\theta}$.*

LEMMA 5. *Suppose f and g are in A, and that $f \circ g=g \circ f$. Suppose that $T(g)=\zeta$ where $|\zeta|=1$. Then $f(\zeta)=\zeta$.*

PROOF. We have $g(0) \in D$, and by the uniqueness of $T(g)$ we have $g(0)\neq 0$. Let S be the segment from 0 to $g(0)$. Then, roughly, what we propose to do is to form a curve γ by joining successive images of S under g^n, and then to use commutativity to show that $f \circ \gamma$ approaches ζ.

For $0\leqq t<1$ let $n(t)$ be the greatest integer less than or equal to $-\log_2(1-t)$. Let $w=g(0)$. Then for $0\leqq t<1$ we define

$$\gamma(t) = g^{n(t)}[2^{n(t)+1}tw - (2^{n(t)+1} - 2)w],$$

and we define $\gamma(1)=\zeta$. It is clear that γ is continuous for $2^{-n-1}<1-t<2^{-n}$, with n a nonnegative integer. We find that $\lim \gamma(t)=g^n(0)=\gamma(1-2^{-n})$ as t approaches $1-2^{-n}$ from above, and that $\lim \gamma(t)=g^{n-1}(w)=g^n(0)$ as t approaches $1-2^{-n}$ from below. Thus γ is continuous, except possibly at 1. But $\lim_{n\to\infty} g^n=\zeta$ uniformly on S, and therefore $\lim_{t\to 1}\gamma(t)=\zeta$, so γ is continuous and terminates at ζ.

Since $f(S)$ is compact, g^n approaches ζ uniformly on $f(S)$. Since $f[g^n(S)]=g^n[f(S)]$, given any neighborhood U of ζ, $f[g^n(S)]\subset U$ for large n. Thus $\lim_{t\to 1} f[\gamma(t)]=\zeta$. By Lindelöf's theorem $\lim_{r\to 1^-} f(r\zeta)=\zeta$, which we write $f(\zeta)=\zeta$.

THEOREM 6. *If $f \in A$ is not a hyperbolic c.a. of D, and if $g \in A$ satisfies $f \circ g=g \circ f$, then $T(f)=T(g)$.*

The proof will use a sequence of lemmas.

LEMMA 7 (CHAIN RULE). *Suppose that f and g are in A, that $|\zeta|=|\eta|=|\tau|=1$, that $f(\zeta)=\eta$, and $g(\eta)=\tau$. Then*

$$\lim_{r\to 1^-}\frac{g[f(r\zeta)] - \tau}{r\zeta - \zeta} = g'(\eta)f'(\zeta).$$

PROOF. Without loss of generality we take $\zeta=\eta=\tau=1$.

If $f'(1)$ and $g'(1)$ are both finite, then $f(r)$ approaches 1 nontangentially, and by Julia's lemma $\lim_{r\to 1^-} f'(r)=f'(1)$ and $\lim_{z\to 1} g'(z)=g'(1)$ when z

approaches 1 nontangentially. The desired result follows from the chain rule for ordinary derivatives. For $r \in D$ real,

$$(1)\qquad \left|\frac{1-g[f(r)]}{1-r}\right| \geqq \frac{1-|g[f(r)]|}{1-r} = \frac{1-|g[f(r)]|}{1-|f(r)|}\cdot\frac{1-|f(r)|}{1-r}.$$

For all $h \in A$ [**1**, p. 25],

$$\frac{1-|h(z)|}{1-|z|} \geqq \frac{1-|h(0)|}{1+|h(0)|} > 0$$

for all $z \in D$, so each factor in the last term of (1) has a positive infimum. It is also true [**1**, p. 27] that if $h'(1)=\infty$, then

$$\lim_{z\to 1}\frac{1-|h(z)|}{1-|z|} = \infty$$

when z approaches 1 in such a way that $h(z)$ approaches 1.

If $f'(1)=\infty$, the desired result follows easily.

Suppose $g'(1)=\infty$ and $f'(1)\neq\infty$. Then $f(r)$ approaches 1 nontangentially, so the first factor in the last term of (1) approaches ∞. Since the second factor has a positive infimum, the desired result follows. This completes the proof of Lemma 7.

As a consequence of Lemma 7, we see that if f and g in A satisfy $f\circ g = g\circ f$ and $|T(f)|=1$, then $T(f^n\circ g^m)$ is $T(f)$ or $T(g)$. For if $T(f^n\circ g^m)=\zeta$, then since $f^n\circ g^m$ commutes with f, we have $|\zeta|=1$, and Lemma 5 shows that $f(\zeta)=g(\zeta)=\zeta$. The derivative of $f^n\circ g^m$ at ζ is $(f'(\zeta))^n(g'(\zeta))^m$ by Lemma 7. In order for this product to be less than or equal to 1, it is necessary that $f'(\zeta)\leqq 1$ or $g'(\zeta)\leqq 1$, which implies $\zeta=T(f)$ or $\zeta=T(g)$.

We shall use the following lemma for the special case $\theta=\phi$.

LEMMA 8. *If $f\in A$, $f(e^{i\theta_1})=e^{i\phi_1}$, and $f(e^{i\theta_2})=e^{i\phi_2}$, where $\theta=\theta_2-\theta_1\neq 0$ (mod 2π) and $\phi=\phi_2-\phi_1\neq 0$ (mod 2π), then*

$$(2)\qquad |f'(e^{i\theta_1})f'(e^{i\theta_2})| \geqq \left[\frac{\sin\phi/2}{\sin\theta/2}\right]^2.$$

This inequality is best possible, and equality obtains only for conformal automorphisms of D.

PROOF. We suppose without loss of generality that $\theta_1=\phi_1=0$. Let $\zeta=e^{i\phi}$ and $\eta=e^{i\theta}$. The set U of conformal automorphisms h of D such that $h(1)=1$ and $h(\zeta)=\eta$ is nonempty, and each such h satisfies $f\circ h(1)=1$ and $f\circ h(\zeta)=\zeta$. By the uniqueness of $T(f\circ h)$ it follows that

$$(3)\qquad |f'(\eta)h'(\zeta)| > 1 \quad\text{or}\quad |f'(1)h'(1)| > 1$$

unless $f \circ h$ is the identity, in which case f is a conformal automorphism of D. We shall show that (3) implies

$$|f'(1)f'(\eta)| > 1/|h'(1)h'(\zeta)|.$$

We observe that the value of $h'(1)h'(\zeta)$ does not depend on $h \in U$, for if h_1 and h_2 are in U, then $h_1h_2^{-1}$ has fixed points 1 and η. Therefore the derivative of $h_1h_2^{-1}$ at η is the reciprocal of its derivative at 1, and this yields

$$h_1'(1)h_1'(\zeta)/h_2'(1)h_2'(\zeta) = 1.$$

When $|a|<1$ let

$$h_a(z) = \frac{1-\bar{a}}{1-a}\frac{z-a}{1-\bar{a}z}.$$

Then

$$h_a'(1) = \frac{1-|a|^2}{(1-a)(1-\bar{a})}$$

which is a continuous function of a for $a \in D$. We have $\lim_{a\to 1} h_a'(1) = \infty$ and $\lim_{a\to w} h_a'(1)=0$ when $|w|=1$ but $w\neq 1$. Moreover $h_a \in U$ provided that

$$\arg(e^{i\phi}-a) - \arg(1-a) = (\phi+\theta)/2 \pmod{2\pi}. \tag{4}$$

The values of a in D which satisfy (4) form a circular arc or line segment terminating at 1 and $e^{i\phi}$. Since this is a connected set, $|h_a'(1)|$ takes all positive real values for $h_a \in U$.

If it were true that $|f'(1)f'(\eta)|<1/|h_a'(1)h_a'(\zeta)|$, then it would follow that

$$|h_a'(1)h_a'(\zeta)| \cdot |f'(\eta)| < 1/|f'(1)|,$$

and since $h_a'(1)h_a'(\zeta)$ is independent of a, we could choose a so that

$$|h_a'(1)h_a'(\zeta)| \cdot |f'(\eta)| < |h_a'(1)| < 1/|f'(1)|.$$

But then we would have

$$|f'(\eta)h_a'(\zeta)| < 1 \quad \text{and} \quad |f'(1)h_a'(1)| < 1,$$

which contradicts (3).

We find that a permissible value of a is

$$\frac{\sin(\theta-\phi)/4}{\sin(\theta+\phi)/4}e^{i\phi/2}.$$

A computation using the observation that $|1-a\zeta|=|1-a|$ and the cosine rule yields

$$\frac{1}{|h_a'(1)h_a'(\zeta)|} = \left[\frac{\sin \phi/2}{\sin \theta/2}\right]^2.$$

This completes the proof of (2).

If $|f'(1)f'(\eta)|=1/|h_a'(1)h_a'(\zeta)|$, then there exists $h\in U$ such that $1/|h'(1)|=|f'(1)|$. Then $|f'(1)h'(1)|=1$ and $|f'(\eta)h'(\zeta)|=1$, so $f=h^{-1}$ is a c.a. of D. If f is a c.a. of D, which satisfies the hypotheses of Lemma 8, then equality holds in (2). This completes the proof of Lemma 8.

We see, in particular, that if $f\in A$, $f(e^{i\theta_1})=e^{i\theta_1}$, and $f(e^{i\theta_2})=e^{i\theta_2}$, then

$$|f'(e^{i\theta_1})f'(e^{i\theta_2})| \geqq 1$$

with equality only if f is a (hyperbolic) c.a. of D.

If the condition $\phi\neq 0$ in Lemma 8 is omitted, then the infimum of the left-hand side of (2) for $f\in A$ is 0.

LEMMA 9. *Given $f'(\zeta)f'(\eta)>1$, $g'(\zeta)g'(\eta)\geqq 1$, $f'(\zeta)\leqq 1$, $f'(\eta)>1$, $g'(\zeta)>1$, $g'(\eta)\leqq 1$, all positive or infinite, there are positive integers n and m such that $(f'(\zeta))^n(g'(\zeta))^m>1$ and $(f'(\eta))^n(g'(\eta))^m>1$.*

(In order to simplify notation we write $\infty>1$.)

PROOF. By taking logarithms one obtains the following equivalent statement.

Given $a+b>0$, $c+d\geqq 0$, $a\leqq 0$, $b>0$, $c>0$, $d\leqq 0$, there are positive integers n and m such that $na+mc>0$ and $nb+md>0$. We prove this form of the statement.

If $d=0$ we simply take $n=1$ and m large. Suppose $d\neq 0$. We find that $a+c|a/c|=0$. Therefore

$$0 < (a + b) + (c + d)\,|a/c| = b + d\,|a/c|,$$

hence $|a/c|<-b/d$, so that $|a/c|<|b/d|$. Let n and m be positive integers such that

$$|a/c| < m/n < |b/d|.$$

Then

$$na + mc = n(a + (m/n)c) > n(a + |a/c|\, c) = 0$$

and

$$nb + md = n(b + (m/n)d) > n(b + |b/d|\, d) = 0.$$

PROOF OF THEOREM 6. The only case in question is that in which $T(f)$ and $T(g)$ are of modulus 1. Assuming that f is not a hyperbolic c.a. of D, that $f\circ g=g\circ f$, and that $T(f)\neq T(g)$, we shall arrive at a contradiction.

Let $T(f)=\zeta$ and $T(g)=\eta$. Since f is not a hyperbolic c.a. of D, and since $f(\eta)=\eta$, Lemma 8 yields $f'(\zeta)f'(\eta)>1$ and $g'(\zeta)g'(\eta)\geqq 1$. We also have $f'(\zeta)\leqq 1$, $f'(\eta)>1$, $g'(\zeta)>1$, and $g'(\eta)\leqq 1$. Now let n and m be as in Lemma 9. Then $T(f^n \circ g^m)\neq\zeta$ and $T(f^n \circ g^m)\neq\eta$, but as observed following the proof of Lemma 7, $T(f^n \circ g^m)$ must be ζ or η. We have arrived at a contradiction, and therefore $T(f)=T(g)$.

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