COMMUTING ANALYTIC FUNCTIONS
WITHOUT FIXED POINTS

DONALD F. BEHAN

Abstract. Let $A$ be the set of nonidentity analytic functions which map the open unit disk into itself. Wolff has shown that the iterates of $f \in A$ converge uniformly on compact sets to a constant $T(f)$, unless $f$ is an elliptic conformal automorphism of the disk. This paper presents a proof that if $f$ and $g$ are in $A$ and commute under composition, and if $f$ is not a hyperbolic conformal automorphism of the disk, then $T(f) = T(g)$. This extends, in a sense, a result of Shields. The proof involves the so-called angular derivative of a function in $A$ at a boundary point of the disk.

Let $D$ be the open unit disk in the complex plane. Let $\bar{D}$ be its closure. Shields [5] has proved the following result.

Theorem 1. If $f$ and $g$ are continuous in $\bar{D}$, analytic in $D$, and map $\bar{D}$ into itself, and if $f \circ g = g \circ f$, then $f$ and $g$ have a common fixed point.

Let $A$ be the set of all analytic functions which map $D$ into $D$, except for the identity function, which we exclude. This paper presents an extension of the result of Shields to the set $A$.

For $f \in A$ we define the iterates of $f$ recursively by $f^1 = f$, and $f^{n+1} = f \circ f^n$ when $n \in \mathbb{Z}^+$. A member of $A$ which maps $D$ univalently onto $D$ will be called a conformal automorphism (c.a.) of $D$. We shall assume that the reader is acquainted with the standard classification of linear fractional transformations as elliptic, hyperbolic, parabolic, or loxodromic, as given in [3, p. 70]. Each c.a. of $D$ is of one of the first three types mentioned. The elliptic transformations yield noneuclidean rotations of $D$ with the hyperbolic metric, while the hyperbolic and parabolic transformations have their fixed points on the boundary of $D$.

Theorem 2 (Wolff [7]). If $f \in A$ is not an elliptic c.a. of $D$, then there is a constant $T(f) \in \bar{D}$ for which $\lim_{n \to \infty} f^n = T(f)$ uniformly on compact sets.

Presented to the Society, January 24, 1971; received by the editors March 1, 1971.


Key words and phrases. Commuting under composition, iteration, fixed point, angular derivative, Julia lemma, chain rule, Lindelöf theorem.
If $T(f)$ is in $D$, then it is a fixed point of $f$; and if $f$ has a fixed point in $D$, it has only one, and that fixed point is $T(f)$. The preceding sentence is still true if we extend the definition of $T$ so that when $f$ is an elliptic c.a. of $D$, then $T(f)$ is the fixed point of $f$ in $D$.

Suppose that $f$ and $g$ are in $A$, that $T(f) \in D$, and that $f \circ g = g \circ f$. Then

$$g[T(f)] = g(f[T(f)]) = f(g[T(f)]),$$

so $g[T(f)]$ is a fixed point for $f$. This implies $T(f) = g[T(f)]$, and consequently $T(g) = T(f)$. We wish to extend this result, as far as possible, to functions $f$ for which $T(f) \notin D$. If two linear fractional transformations have the same set of fixed points, then they commute [3, p. 72]. There exist pairs $f$, $g$ of hyperbolic conformal automorphisms of $D$ which have a common set of fixed points, but for which $T(f) \neq T(g)$. For example

$$f(z) = \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z} \quad \text{and} \quad g(z) = \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z}$$

defines such a pair. Thus it is not true in general that commuting members of $A$ satisfy $T(f) = T(g)$, but we shall show that the only exceptional cases involve pairs of hyperbolic conformal automorphisms of $D$.

The only cases in which linear fractional transformations commute without having the same fixed points involve certain pairs of elliptic transformations which cannot occur as pairs in $A$. Thus two conformal automorphisms of $D$ commute if and only if they have the same fixed points.

For ease of notation when $f \in A$ and $|\zeta| = 1$ we shall write $f(\zeta) = \eta$ to mean $\lim_{r \to 1^-} f(r\zeta) = \eta$, and in that case we write $f'(\zeta) = u$ to mean

$$\lim_{r \to 1^-} \frac{f(r\zeta) - \eta}{r\zeta - \zeta} = u.$$

We shall have several occasions to use Julia's lemma, which we state in a form which is essentially that of [4, p. 58].

**Theorem 3.** If $|\zeta| = |\eta| = 1$, and $f \in A$ satisfies $f(\zeta) = \eta$, then there is an extended real number $c$, with $0 < c \leq \infty$, such that

$$\lim_{z \to \zeta} \left| \frac{f(z) - \eta}{z - \zeta} \right| = c$$

as $z$ approaches $\zeta$ through any open triangle in $D$ with vertex at $\zeta$. If $c$ is finite then $f'(\zeta) = cn/\zeta$.

If $c$ is finite, then it is also true that $\lim_{z \to \zeta} f'(z) = f'(\zeta)$ when $z$ approaches $\zeta$ as in Theorem 3.
If $f \in A$ and $|\zeta|=1$, then $T(f)=\zeta$ if and only if $f(\zeta)=\zeta$ and $f'(\zeta)\leq 1$. (See [6] for example.)

We shall use the following result of Lindelöf [2, p. 19].

**Theorem 4.** Let $f$ be analytic and bounded in $D$. If $f(z)$ approaches $w$ as $z$ approaches $e^{i\theta}$ along some curve $\gamma$ lying in $D$ except for its terminal point at $e^{i\theta}$, then $f(z)$ approaches $w$ uniformly as $z$ approaches $e^{i\theta}$ in any open triangle in $D$ with vertex $e^{i\theta}$.

**Lemma 5.** Suppose $f$ and $g$ are in $A$, and that $f \circ g = g \circ f$. Suppose that $T(g) = \zeta$ where $|\zeta|=1$. Then $f(\zeta) = \zeta$.

**Proof.** We have $g(0) \in D$, and by the uniqueness of $T(g)$ we have $g(0) \neq 0$. Let $S$ be the segment from 0 to $g(0)$. Then, roughly, what we propose to do is to form a curve $\gamma$ by joining successive images of $S$ under $g^n$, and then to use commutativity to show that $f \circ \gamma$ approaches $\zeta$.

For $0 \leq t < 1$ let $n(t)$ be the greatest integer less than or equal to $-\log_2(1-t)$. Let $w = g(0)$. Then for $0 \leq t < 1$ we define

$$
\gamma(t) = g^{n(t)}[2^{n(t)+1}w - (2^{n(t)+1} - 2)w],
$$

and we define $\gamma(1) = \zeta$. It is clear that $\gamma$ is continuous for $2^{-n-1} < 1-t < 2^{-n}$, with $n$ a nonnegative integer. We find that $\lim_{t \to 1-} \gamma(t) = \gamma(1-2^{-n})$ as $t$ approaches $1-2^{-n}$ from above, and that $\lim_{t \to 1+} \gamma(t) = \gamma(1-2^{-n})$ as $t$ approaches $1-2^{-n}$ from below. Thus $\gamma$ is continuous, except possibly at 1.

But $\lim_{t \to 1} g^n = \zeta$ uniformly on $S$, and therefore $\lim_{t \to 1} \gamma(t) = \zeta$, so $\gamma$ is continuous and terminates at $\zeta$.

Since $f(S)$ is compact, $g^n$ approaches $\zeta$ uniformly on $f(S)$. Since $f[g^n(S)] = g^n[f(S)]$, given any neighborhood $U$ of $\zeta$, $f[g^n(S)] \subseteq U$ for large $n$. Thus $\lim_{t \to 1} f[\gamma(t)] = \zeta$. By Lindelöf’s theorem $\lim_{t \to 1} f(r^\zeta) = \zeta$, which we write $f(\zeta) = \zeta$.

**Theorem 6.** If $f \in A$ is not a hyperbolic c.a. of $D$, and if $g \in A$ satisfies $f \circ g = g \circ f$, then $T(f) = T(g)$.

The proof will use a sequence of lemmas.

**Lemma 7 (Chain Rule).** Suppose that $f$ and $g$ are in $A$, that $|\zeta|=|\eta|=|\tau|=1$, that $f(\zeta) = \eta$, and $g(\eta) = \tau$. Then

$$
\lim_{r \to 1-} \frac{g[f(\zeta)] - \tau}{r^\zeta - \zeta} = g'(\eta)f'(\zeta).
$$

**Proof.** Without loss of generality we take $\zeta = \eta = \tau = 1$.

If $f'(1)$ and $g'(1)$ are both finite, then $f(r)$ approaches 1 nontangentially, and by Julia's lemma $\lim_{r \to 1-} f'(r) = f'(1)$ and $\lim_{z \to 1} g'(z) = g'(1)$ when $z$
approaches 1 nontangentially. The desired result follows from the chain rule for ordinary derivatives. For \( r \in D \) real,

\[
\frac{1 - g[f(r)]}{1 - r} \geq \frac{1 - |g[f(r)]|}{1 - |f(r)|} = 1 - \frac{|g[f(r)]|}{1 - |f(r)|} = \frac{1 - |f(r)|}{1 - r}.
\]

For all \( h \in A \) [1, p. 25],

\[
\frac{1 - |h(z)|}{1 - |z|} \geq \frac{1 - |h(0)|}{1 + |h(0)|} > 0
\]

for all \( z \in D \), so each factor in the last term of (1) has a positive infimum. It is also true [1, p. 27] that if \( A'(1) = \infty \), then

\[
\lim_{z \to 1} \frac{1 - |h(z)|}{1 - |z|} = \infty
\]

when \( z \) approaches 1 in such a way that \( h(z) \) approaches 1.

If \( f'(1) = \infty \), the desired result follows easily.

Suppose \( g'(1) = \infty \) and \( f'(1) \neq \infty \). Then \( f(r) \) approaches 1 nontangentially, so the first factor in the last term of (1) approaches \( \infty \). Since the second factor has a positive infimum, the desired result follows. This completes the proof of Lemma 7.

As a consequence of Lemma 7, we see that if \( f \) and \( g \) in \( A \) satisfy \( f \circ g = g \circ f \), then \( T(f) = T(g) \). For if \( T(f^n \circ g^m) = \zeta \), then since \( f^n \circ g^m \) commutes with \( f \), we have \( |\zeta| = 1 \), and Lemma 5 shows that \( f(\zeta) = g(\zeta) = \zeta \). The derivative of \( f^n \circ g^m \) at \( \zeta \) is \( (f'(\zeta))^n (g'(\zeta))^m \) by Lemma 7. In order for this product to be less than or equal to 1, it is necessary that \( f'(\zeta) \leq 1 \) or \( g'(\zeta) \leq 1 \), which implies \( \zeta = T(f) \) or \( \zeta = T(g) \).

We shall use the following lemma for the special case \( \theta = \phi \).

**Lemma 8.** If \( f \in A \), \( f(e^{i\theta_1}) = e^{i\phi_1} \), and \( f(e^{i\theta_2}) = e^{i\phi_2} \), where \( \theta = \theta_2 - \theta_1 \neq 0 \) (mod \( 2\pi \)) and \( \phi = \phi_2 - \phi_1 \neq 0 \) (mod \( 2\pi \)), then

\[
|f'(e^{i\phi_1})f'(e^{i\phi_2})| \geq \frac{|\sin \phi/2|^2}{|\sin \theta/2|^2}.
\]

This inequality is best possible, and equality obtains only for conformal automorphisms of \( D \).

**Proof.** We suppose without loss of generality that \( \theta_1 = \phi_1 = 0 \). Let \( \zeta = e^{i\theta} \) and \( \eta = e^{i\phi} \). The set \( U \) of conformal automorphisms \( h \) of \( D \) such that \( h(1) = 1 \) and \( h(\zeta) = \eta \) is nonempty, and each such \( h \) satisfies \( f \circ h(1) = 1 \) and \( f \circ h(\zeta) = \zeta \). By the uniqueness of \( T(f \circ h) \) it follows that

\[
|f'(\eta)h'(\zeta)| > 1 \quad \text{or} \quad |f'(1)h'(1)| > 1
\]
unless \( f \circ h \) is the identity, in which case \( f \) is a conformal automorphism of \( D \). We shall show that (3) implies

\[ |f''(1)f''(\eta)| > 1/|h'(1)h'(\xi)|. \]

We observe that the value of \( h'(1)h'(\xi) \) does not depend on \( h \in U \), for if \( h_1 \) and \( h_2 \) are in \( U \), then \( h_1h_2^{-1} \) has fixed points 1 and \( \eta \). Therefore the derivative of \( h_1h_2^{-1} \) at \( \eta \) is the reciprocal of its derivative at 1, and this yields

\[ h_1'(1)h_1'(\xi)/h_2'(1)h_2'(\xi) = 1. \]

When \(|a|<1\) let

\[ h_a(z) = \frac{1 - \bar{a}z - a}{1 - a - \bar{a}z}. \]

Then

\[ h_a'(1) = \frac{1 - |a|^2}{(1 - a)(1 - \bar{a})} \]

which is a continuous function of \( a \) for \( a \in D \). We have \( \lim_{a \to 1} h_a'(1) = \infty \) and \( \lim_{a \to w} h_a'(1) = 0 \) when \(|w| = 1\) but \( w \neq 1 \). Moreover \( h_a \in U \) provided that

(4) \[ \arg(e^{i\phi} - a) - \arg(1 - a) = (\phi + \theta)/2 \pmod{2\pi}. \]

The values of \( a \) in \( D \) which satisfy (4) form a circular arc or line segment terminating at 1 and \( e^{i\phi} \). Since this is a connected set, \(|h_a'(1)|\) takes all positive real values for \( h_a \in U \).

If it were true that \(|f''(1)f''(\eta)| < 1/|h_a'(1)h_a'(\xi)|\), then it would follow that

\[ |h_a'(1)h_a'(\xi)| \cdot |f''(\eta)| < 1/|f''(1)|, \]

and since \( h_a'(1)h_a'(\xi) \) is independent of \( a \), we could choose \( a \) so that

\[ |h_a'(1)h_a'(\xi)| \cdot |f''(\eta)| < |h_a'(1)| < 1/|f''(1)|. \]

But then we would have

\[ |f''(\eta)h_a'(\xi)| < 1 \quad \text{and} \quad |f''(1)h_a'(1)| < 1, \]

which contradicts (3).

We find that a permissible value of \( a \) is

\[ \sin(\theta - \phi)/4 e^{i\phi/2} \]

\[ \sin(\theta + \phi)/4. \]

A computation using the observation that \(|1-a\xi| = |1-a|\) and the cosine
rule yields
\[ \frac{1}{|h_2'(1)h_2'(\zeta)|} = \left[ \frac{\sin \phi/2}{\sin \theta/2} \right]^2. \]

This completes the proof of (2).

If \(|f'(1)f'(\eta)| = 1/|h_2'(1)h_2'(\zeta)|\), then there exists \( h \in U \) such that \( 1/|h'(1)| = |f'(1)| \) and \( |f'(\eta)h'(\zeta)| = 1 \), so \( f = h^{-1} \) is a c.a. of \( D \). If \( f \) is a c.a. of \( D \), which satisfies the hypotheses of Lemma 8, then equality holds in (2). This completes the proof of Lemma 8.

We see, in particular, that if \( f \in A, f(e^{i\theta_2}) = e^{i\theta_2} \), and \( f(e^{i\theta_3}) = e^{i\theta_3} \), then
\[ |f'(e^{i\theta_2})f'(e^{i\theta_3})| \geq 1 \]
with equality only if \( f \) is a (hyperbolic) c.a. of \( D \).

If the condition \( \phi \neq 0 \) in Lemma 8 is omitted, then the infimum of the left-hand side of (2) for \( f \in A \) is 0.

**Lemma 9.** Given \( f'(\zeta)f'(\eta) > 1, g'(\zeta)g'(\eta) > 1, f'(\zeta) \leq 1, f'(\eta) > 1, g'(\zeta) > 1, g'(\eta) \leq 1 \), all positive or infinite, there are positive integers \( n \) and \( m \) such that \( (f'(\zeta))^n(g'(\zeta))^m > 1 \) and \( (f'(\eta))^n(g'(\eta))^m > 1 \).

(In order to simplify notation we write \( \infty > 1 \).)

**Proof.** By taking logarithms one obtains the following equivalent statement.

Given \( a + b, c + d > 0, a, b > 0, c > 0, d > 0 \), there are positive integers \( n \) and \( m \) such that \( na + mc > 0 \) and \( nb + md > 0 \). We prove this form of the statement.

If \( d = 0 \) we simply take \( n = 1 \) and \( m \) large. Suppose \( d \neq 0 \). We find that \( a + c|a/c| = 0 \). Therefore
\[ 0 < (a + b) + (c + d)|a/c| = b + d|a/c|, \]

hence \( |a/c| < -b/d \), so that \( |a/c| < |b/d| \). Let \( n \) and \( m \) be positive integers such that
\[ |a/c| < m/n < |b/d|. \]

Then
\[ na + mc = n(a + (m/n)c) > n(a + |a/c| c) = 0 \]
and
\[ nb + md = n(b + (m/n)d) > n(b + |b/d| d) = 0. \]

**Proof of Theorem 6.** The only case in question is that in which \( T(f) \) and \( T(g) \) are of modulus 1. Assuming that \( f \) is not a hyperbolic c.a. of \( D \), that \( f \circ g = g \circ f \), and that \( T(f) \neq T(g) \), we shall arrive at a contradiction.

Let \( T(f) = \zeta \) and \( T(g) = \eta \). Since \( f \) is not a hyperbolic c.a. of \( D \), and since \( f(\eta) = \eta \), Lemma 8 yields \( f'(\zeta)f'(\eta) > 1 \) and \( g'(\zeta)g'(\eta) \geq 1 \). We also
have $f'(\zeta) \leq 1$, $f'(\eta) > 1$, $g'(\zeta) > 1$, and $g'(\eta) \leq 1$. Now let $n$ and $m$ be as in Lemma 9. Then $T(f^n \circ g^m) \neq \zeta$ and $T(f^n \circ g^m) \neq \eta$, but as observed following the proof of Lemma 7, $T(f^n \circ g^m)$ must be $\zeta$ or $\eta$. We have arrived at a contradiction, and therefore $T(f) = T(g)$.

REFERENCES


Department of Mathematics, Union College, Schenectady, New York 12308

Current address: Farm Family Life Insurance Co., Box 656, Albany, New York 12201