COMPACT RIEMANN SURFACES WITH CONFORMAL INVOLUTIONS

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Abstract. In this paper surfaces which have conformal involutions are characterized by their period matrices.

Let $S$ be a compact Riemann surface of genus $g$, $g \neq 0$, and $j$ a conformal involution on $S$ with $k$ fixed points, $k \neq 0$. Let $S'$ be the factor surface and $g'$ the genus of $S'$. Then $g = 2g' + k/2 - 1$ by the Riemann-Hurwitz relation, so that $k$ is even.

Proposition 1. $S$ has a canonical homology basis consisting of homology classes of curves of the form

\begin{equation}
(1) \quad a_1, \cdots, a_{g'}, j(a_1), \cdots, j(a_{g'}), c_1, \cdots, c_{k/2-1},
\end{equation}

\begin{equation}
(2) \quad b_1, \cdots, b_{g'}, j(b_1), \cdots, j(b_{g'}), d_1, \cdots, d_{k/2-1},
\end{equation}

where $j(c_i) \sim -c_i$ and $j(d_i) \sim -d_i$ and where $\sim$ denotes homologous.

Proof. Let $P_1, \cdots, P_k$ be the fixed points of $j$ on $S$ and also their images on $S'$. Let $(a, b)$ be $2g'$ curves whose homology classes give a canonical homology basis for $S'$. Let $n_i$ be a path on $S'$ from $P_i$ to $P_{i+1}$ for all $i$ which does not intersect any curve in $(a, b)$. Then it can be shown that $S$ is two copies of $S'$ slit along the $n_i$ for $i$ odd and pasted together appropriately for some choice of slits $n_i$. If $c$ is any path on $S'$, let $\hat{c}$ be a lifting to $S$. Let $N_i = \bar{n}_i - j(\bar{n}_i)$ for $i = 1, 2, \cdots, k - 1$. Let $d_i = N_i$ if $i$ is even; $d_i = N_i$; and define $d_{2i+1} = d_{2i-1} - N_{2i+1}$ inductively. Since $j(N_i) = -N_i$, $j(d_i) = -d_i$ for all $i$. Let $c_i = d_i$ and $d_{i+1} = d_{2i+1}$. Consider the set:

\begin{align*}
\tilde{a}_1, \cdots, \tilde{a}_{g'}, j(\tilde{a}_1), \cdots, j(\tilde{a}_{g'}), c_1, \cdots, c_{k/2-1}, \\
\tilde{b}_1, \cdots, \tilde{b}_{g'}, j(\tilde{b}_1), \cdots, j(\tilde{b}_{g'}), d_1, \cdots, d_{k/2-1}.
\end{align*}

Let $A \times B$ denote the intersection number of $A$ and $B$. Since $N_i \times N_{i+1} = 1$ for all $i$, we have: $\tilde{a}_i \times \tilde{b}_j = \delta_{ij}$; $\tilde{a}_i \times j(\tilde{b}_j) = 0$; $\tilde{a}_i \times c_j = 0$; $\tilde{a}_i \times d_j = 0$; $\tilde{a}_i \times j(\tilde{a}_j) = 0$; $\tilde{b}_i \times j(\tilde{b}_j) = 0$ and $c_i \times d_j = \delta_{ij}$ for all relevant $i$ and $j$. These

Received by the editors January 12, 1972.


1 This paper contains a part of the author's Ph.D. thesis written at Columbia University under the direction of Professor Lipman Bers whom the author wishes to thank for his help and encouragement.

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intersection numbers can be used to show that these are $2g$ homologously
independent curves. Therefore, their homology classes must form a
canonical homology basis for $S$.

**Definition.** A canonical homology basis of the form in Proposition 1
will be called a homology basis adapted to the involution $j$.

**Proposition 2.** Assume further that $k \neq 4$. Then $S$ has a conformal
involution with $k$ fixed points if and only if it has a period matrix of the form
$(I, Z)$ where

$$
Z = \begin{pmatrix}
A & B & C \\
B & A & -C \\
^tC & -^tC & K
\end{pmatrix}
$$

where $A$ and $B$ are $g' \times g'$ symmetric, $C$ is $g' \times (k/2-1)$, and $K$ is $(k/2-1) \times
(k/2-1)$ symmetric. Further, $(I, (A+B))$ is a period matrix for $S'$.

**Proof.** Pick a homology basis adapted to $j$. Let $w_i$ be the differential
which has period 1 with respect to $a_i$ and period zero with respect to all
other curves in line 1. Let $j^*$ be the map which $j$ induces on differentials.
Then $j^*(w_i)$ has period 1 with respect to $j(a_i)$ and period zero with respect
to all other curves on line 1. Let $w_c_i$ be the differential with period 1 with
respect to $c_i$ and period zero with respect to all other curves in line 1.
Form the period matrix of $S$ with respect to these bases and use the fact
that $\int_{j(z)} j^*(y) = \int_x y$ for any $x$ and $y$ to simplify the period matrix to the
desired form.

Finally since $(a, b)$ is a canonical homology basis for $S'$ and since
$(w_i+j^*(w_i))$ as a $j$-invariant holomorphic differential defines a differential
on $S'$, $(I, (A+B))$ is a period matrix for $S'$.

To prove the converse we let $J_{g',k}$ be the matrix of the action of $j$ on a
homology basis adapted to $j$ if $j$ is a conformal involution with $k$ fixed
points. Then $Z$ is of the form shown in Proposition 2 if and only if
$J_{g',k}(Z) = Z$, where $J_{g',k}$ now acts as an element of the (inhomogeneous)
Siegel modular group. Indeed, if $J_{g',k}$, viewed as an element of $S_g(g, Z)$,
acts on the homology basis, viewed as a column vector whose elements are
arranged in the order of (1) then (2) of Proposition 1, then

$$
J_{g',k} = \begin{pmatrix}
M & 0 \\
0 & M
\end{pmatrix},
$$

where $M$ is $g \times g$ and

$$
M = \begin{pmatrix}
0 & I_{g'} & 0 \\
I_{g'} & 0 & 0 \\
0 & 0 & -I_{k/2-1}
\end{pmatrix},
$$
$J_{\varphi', k}(Z) = MZM$

from which the form of $Z$ follows.

$J_{\varphi', k}(Z) = Z$ implies (see [2, p. 28]) $S$ has a conformal involution whose action on homology is either given by $J_{\varphi', k}$ or $-J_{\varphi', k}$. If the latter occurs, apply the main result of [1] to conclude that either $g=0$, contrary to assumption, or $k$ is less than 4.

In either case, $S$ has a conformal involution and we can apply either the Atiyah-Singer index theorem or the Lefschetz fixed point formula to show that the number of fixed points is $2 - \text{tr} J_{\varphi', k} = k$ in the one case and $2 - \text{tr}(-J_{\varphi', k}) = 4 - k$ in the other case. If $k$ is greater than 4, the first case occurs; $k=4$ is excluded; and if $k=2$, whichever case occurs, the involution has 2 fixed points.

REFERENCES
