COMPACT RIEMANN SURFACES WITH CONFORMAL INVOLUTIONS\textsuperscript{1}

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Abstract. In this paper surfaces which have conformal involutions are characterized by their period matrices.

Let $S$ be a compact Riemann surface of genus $g$, $g\neq 0$, and $j$ a conformal involution on $S$ with $k$ fixed points, $k\neq 0$. Let $S'$ be the factor surface and $g'$ the genus of $S'$. Then $g=2g'+k/2-1$ by the Riemann-Hurwitz relation, so that $k$ is even.

Proposition 1. $S$ has a canonical homology basis consisting of homology classes of curves of the form

\begin{align*}
(1) & \quad a_1, \ldots, a_g, j(a_1), \ldots, j(a_g), c_1, \ldots, c_{k/2-1}, \\
(2) & \quad b_1, \ldots, b_g, j(b_1), \ldots, j(b_g), d_1, \ldots, d_{k/2-1},
\end{align*}

where $j(c_i)\sim -c_i$ and $j(d_i)\sim -d_i$ and where $\sim$ denotes homologous.

Proof. Let $P_1, \ldots, P_k$ be the fixed points of $j$ on $S$ and also their images on $S'$. Let $(a, b)$ be $2g$ curves whose homology classes give a canonical homology basis for $S'$. Let $n_i$ be a path on $S'$ from $P_i$ to $P_{i+1}$ for all $i$ which does not intersect any curve in $(a, b)$. Then it can be shown that $S$ is two copies of $S'$ slit along the $n_i$ for $i$ odd and pasted together appropriately for some choice of slits $n_i$. If $c$ is any path on $S'$, let $\tilde{c}$ be a lifting to $S$. Let $N_i=n_i-j(n_i)$ for $i=1, 2, \ldots, k-1$. Let $d_i=N_i$ if $i$ is even; $d_i=N_i$; and define $d_{2i+1}=d_{2i-1}-N_{2i+1}$ inductively. Since $j(N_i)=-N_i$, $j(d_i)=-d_i$ for all $i$. Let $c_i=d_{2i}$ and $d_{i+1}=d_{2i+1}$. Consider the set:

\begin{align*}
\tilde{a}_1, \ldots, \tilde{a}_g, j(\tilde{a}_1), \ldots, j(\tilde{a}_g), c_1, \ldots, c_{k/2-1}, \\
\tilde{b}_1, \ldots, \tilde{b}_g, j(\tilde{b}_1), \ldots, j(\tilde{b}_g), d_1, \ldots, d_{k/2-1}.
\end{align*}

Let $A \times B$ denote the intersection number of $A$ and $B$. Since $N_i \times N_{i+1}=1$ for all $i$, we have: $\tilde{a}_i \times \tilde{b}_j=\delta_{ij}$; $\tilde{a}_i \times j(\tilde{b}_j)=0$; $\tilde{a}_i \times c_j=0$; $\tilde{a}_i \times d_j=0$; $\tilde{a}_i \times j(\tilde{a}_j)=0$; $\tilde{b}_i \times j(\tilde{b}_j)=0$ and $c_i \times d_j=\delta_{ij}$ for all relevant $i$ and $j$. These

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intersection numbers can be used to show that these are 2g homologously independent curves. Therefore, their homology classes must form a canonical homology basis for S.

**Definition.** A canonical homology basis of the form in Proposition 1 will be called a *homology basis adapted to the involution j*.

**Proposition 2.** Assume further that \( k \neq 4 \). Then S has a conformal involution with k fixed points if and only if it has a period matrix of the form \((I, Z)\) where

\[
Z = \begin{pmatrix}
A & B & C \\
B & A & -C \\
-C & C & K
\end{pmatrix}
\]

where A and B are \( g' \times g' \) symmetric, \( C \) is \( g' \times (k/2 - 1) \), and \( K \) is \( (k/2 - 1) \times (k/2 - 1) \) symmetric. Further, \((I, (A+B))\) is a period matrix for \( S' \).

**Proof.** Pick a homology basis adapted to j. Let \( w_i \) be the differential which has period 1 with respect to \( a_i \) and period zero with respect to all other curves in line 1. Let \( j^* \) be the map which j induces on differentials. Then \( j^*(w_i) \) has period 1 with respect to \( j(a_i) \) and period zero with respect to all other curves on line 1. Let \( w_{c_i} \) be the differential with period 1 with respect to \( c_i \) and period zero with respect to all other curves in line 1. Form the period matrix of S with respect to these bases and use the fact that \( \int_{j(z)} j^*(y) = \int_x y \) for any \( x \) and \( y \) to simplify the period matrix to the desired form.

Finally since \((a, b)\) is a canonical homology basis for \( S' \) and since \((w_i + j^*(w_i))\) as a j-invariant holomorphic differential defines a differential on \( S' \), \((I, (A+B))\) is a period matrix for \( S' \).

To prove the converse we let \( J_{g',k} \) be the matrix of the action of j on a homology basis adapted to j if j is a conformal involution with k fixed points. Then Z is of the form shown in Proposition 2 if and only if \( J_{g',k}(Z) = Z \), where \( J_{g',k} \) now acts as an element of the (inhomogeneous) Siegel modular group. Indeed, if \( J_{g',k} \), viewed as an element of \( S_g(g, Z) \), acts on the homology basis, viewed as a column vector whose elements are arranged in the order of (1) then (2) of Proposition 1, then

\[
J_{g',k} = \begin{pmatrix}
M & 0 \\
0 & M
\end{pmatrix},
\]

where \( M \) is \( g \times g \) and

\[
M = \begin{pmatrix}
I_{g'} & 0 & 0 \\
0 & I_{g'} & 0 \\
0 & 0 & -I_{k/2-1}
\end{pmatrix},
\]
$I_{g',k/2-1}$ being the indicated identity matrices; and then

$$J_{g',k}(Z) = MZM$$

from which the form of $Z$ follows.

$J_{g',k}(Z)=Z$ implies (see [2, p. 28]) $S$ has a conformal involution whose action on homology is either given by $J_{g',k}$ or $-J_{g',k}$. If the latter occurs, apply the main result of [1] to conclude that either $g=0$, contrary to assumption, or $k$ is less than 4.

In either case, $S$ has a conformal involution and we can apply either the Atiyah-Singer index theorem or the Lefschetz fixed point formula to show that the number of fixed points is $2 - \text{tr} J_{g',k} = k$ in the one case and $2 - \text{tr}(-J_{g',k}) = 4 - k$ in the other case. If $k$ is greater than 4, the first case occurs; $k=4$ is excluded; and if $k=2$, whichever case occurs, the involution has 2 fixed points.

References


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