PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 1, January 1973

A MODEL FOR CLOSED ORIENTABLE 3-MANIFOLDS OF GENUS 1

J. A. NARVARTE

ABSTRACT. In this paper we obtain a rectangular model for each closed orientable 3-manifold of genus 1 by identifications of the unit cube via unimodular plane involutions. The desirability of the model is indicated with regard to the fibring of such manifolds by circles, together with the development of a corresponding model for the connected sum of two such manifolds.

1. **Introduction.** In [**1**], a "rectangular" model $\mathscr{K}$ for S^3 is obtained from the unit cube via certain boundary identifications and is used to show how the S^1-fibrings (Seifert fibrings) of S^3 arise in a natural way from the Hilbert modular functions. Here the main emphasis is on the fibrings, and the author quite naturally chooses to show that the closed (compact, without boundary) orientable 3-manifold $\mathscr{K}$ is in fact S^3 by observing that $\mathscr{K}$ has an S^1-fibring with base space S^2. The need to generalize $\mathscr{K}$ to include the lens spaces $L(p, q)$ is suggested in [**1**] and is therefore the motivation for this paper.

It is well known that each closed orientable 3-manifold M of genus 1 is either a lens space or the duplication $S^2 \times S^1$. The purpose of this paper is to obtain the corresponding model $\mathscr{K}$ for each such M from the unit cube by identifying coordinates x, y each modulo 1 and the remaining faces each via a unimodular plane involution. An indication as to the desirability of the model for visualizing the S^1-fibrings of M is included in the final section.

2. **Remarks on unimodular plane involutions.** By unimodular plane involution is meant a linear operator T_0 of the plane with integral coefficients which is involutory, and whose coefficients have determinant -1. In general, T_0 is of the form

$$T_0(x, y) = (a_0x + b_0y, c_0x - a_0y).$$

A linear form s_0 which is invariant under T_0 is given by $s_0=(a_0+1)x+b_0y$ (or $s_0=c_0x+2y$ in case $a_0+1=b_0=0$), and if d_0 is the gcd of the coefficients

Received by the editors May 5, 1972.

AMS (MOS) subject classifications (1970). Primary 57A10; Secondary 55F55.

Key words and phrases. Genus 1, rectangular model, unimodular plane involution, fibre space.

of s_0, then $S_0=s_0/d_0$ is a generator of the submodule of T_0-invariant linear forms with integral coefficients. Similarly, a linear form s_0' which is anti-invariant under T_0 is given by $s_0'=(a_0-1)x+b_0y$ (or $s_0'=c_0x-2y$ in case $a_0-1=b_0=0$), and if d_0' is the gcd of its coefficients, then $S_0'=s_0'/d_0'$ is a generator of the submodule of T_0-anti-invariant linear forms.

Let $\delta_0=\det(S_0, S_0')$ denote the determinant of the coefficients of S_0 and S_0'. By choosing the sign of S_0' accordingly, δ_0 can be taken to be positive, and by direct calculation, $\delta_0=2$ or 1, depending on whether d_0' divides or does not divide b_0/d_0.

Let $\bar{S}_0$ denote S_0' whenever $\delta_0=1$ and $(S_0+S_0')/2$ otherwise. In either case, a change of coordinates to the new coordinates S_0, $\bar{S}_0$ is unimodular, and (replacing S_0, $\bar{S}_0$ by x, y) we have

LEMMA 1. *By a suitable change of coordinates, each unimodular plane involution can be written in the form*

$$T_0(x, y) = (x, (\delta_0 - 1)x - y), \quad \textit{for } \delta_0 = 1 \text{ or } 2.$$

Since the involutory property of T_0 is invariant under the identification of the plane which identifies coordinates x, y each modulo 1, then T_0 induces a "folding" of the unit square which identifies each point of the square with its image under T_0, modulo the square. By a change of coordinates (with new coordinates S_0, $\bar{S}_0$), write T_0 in the form of Lemma 1. Then for $\delta_0=1$, $T_0(x, y)=(x, -y)$ with invariant and anti-invariant linear forms $S_0=x$ and $S_0'=y$ respectively, so that T_0 has invariant lines $x=$ const. and anti-invariant lines $y=$const. (T_0 maps the line $x=c$ onto itself and the line $y=c$ onto $y=-c$.) For $\delta_0=2$, $T_0(x, y)=(x, x-y)$ with $S_0=x$ and $S_0'=-x+2y$, so that T_0 has invariant lines $x=$const. and anti-invariant lines $-x+2y=$const. In either case, reinserting S_0, $\bar{S}_0$ for x, y we have

LEMMA 2. *T_0 induces a folding of the unit square which identifies points along the lines $S_0=$const. and symmetrically through the ramification line $S_0'=0$, with coordinates x, y each taken modulo* 1.

Let V be the half-cube $\{0\leqq x, y\leqq 1, 0\leqq z\leqq 1/2\}$, and V_0 the compact orientable 3-manifold obtained from V by identifying coordinates x, y each modulo 1, and the face $z=1/2$ via T_0 as in Lemma 2. That V_0 is a toroid (solid torus of genus 1) with boundary π_0 (the $z=0$ face of V) is apparent by changing coordinates to S_0, $\bar{S}_0$. Then T_0 takes the form of Lemma 1, and in either case ($\delta_0=1$ or 2) the simple closed curve $x=0$ on π_0 is nullhomotopic within V_0. Moreover, the required homotopy, given by $H_t(0, y, z)=(0, y, z-t(z-1/2))$, deforms the $x=0$ face of V_0 to a point, since $H_1(0, y, z)=(0, y, 1/2)$ is self-cancelling under T_0. Hence the $x=0$

face of V_0 is topologically a 2-disc, and since V_0 is the product of its $x=0$ face and the simple closed curve $y=0$ on π_0, V_0 is a toroid. So, in terms of S_0, $\bar{S}_0$ we have

LEMMA 3. *V_0 is a toroid with meridian $S_0=0$ and longitude $\bar{S}_0=0$.*

3. **Genus 1 model.** If M is the closed orientable 3-manifold of genus 1 obtained from toroids V_0 and V_1 by identifying their boundaries according to an orientation reversing homeomorphism, and if V_i has meridian M_i and longitude B_i, $i=0, 1$, then if $M_1 \sim \pm M_0$ ($\sim$means homotopic) on the common boundary π, M is clearly the duplication $S^2 \times S^1$. If $M_1 \sim pB_0 + qM_0$ on π, with $p>0$ and q relatively prime integers, M is the lens space $L(p, q)$ [3, p. 554]. Let T_0 and T_1 be unimodular plane involutions with T_0 in the form of Lemma 1. Denote by $\mathscr{K}_1$ the closed orientable 3-manifold obtained from the unit cube by identifying coordinates x, y each modulo 1 and the faces $z=0$ and $z=1$ via T_0 and T_1 respectively. The plane $z=1/2$ decomposes $\mathscr{K}_1$ into toroids V_0 and V_1 with common boundary π (the $z=1/2$ cross-section), so that $\mathscr{K}_1$ is a genus 1 manifold.

By Lemma 3, V_0 has meridian $x=0$ and longitude $y=0$ on π. If $b_1 \neq 0$, the invariant linear form for T_1 is given by $S_1=[(a_1+1)x+b_1y]/d_1$, so that V_1 has meridian $(a_1+1)/d_1x+b_1/d_1y=0$ on π. If $b_1=0$, then $a_1=\pm 1$. For $a_1=-1$, $S_1=(c_1x+2y)/d_1$, so that V_1 has meridian $c_1/d_1x+2/d_1y=0$ on π, and for $a_1=+1$, $T_1(x, y)=(x, c_1x-y)$ with $S_1=x$, so that V_1 has meridian $x=0$ on π. Combining the above remarks we have

THEOREM. *If $p \geqq 0$ and q are relatively prime integers and T_1 a unimodular plane involution with invariant linear form $S_1=qx+py$, then $\mathscr{K}_1$ is the duplication $S^2 \times S^1$ whenever $p=0$ and the lens space $L(p, q)$ otherwise.*

A partial converse can be given by

COROLLARY 1. *Each lens space $L(p, q)$ has a corresponding model $\mathscr{K}_1$, where T_0 is given by $T_0(x, y)=(x, -y)$ and T_1 by $T_1(x, y)=((dq-1)x+dpy, cqx-(dq-1)y)$, with c, d integral solutions to $cp+dq=2$.*

Let T_0 and T_1 be arbitrary unimodular plane involutions, and $\mathscr{K}$ be obtained from the unit cube via T_0 and T_1 as before. We say that T_0 and T_1 are *p-compatible* if $\det(S_0, S_1)=p \geqq 0$. ($p \geqq 0$ is guaranteed by choosing the sign of S_1 appropriately.) Since p-compatibility is invariant under unimodular change of coordinates, we have

COROLLARY 2. *If T_0 and T_1 are 0-compatible, $\mathscr{K}=S^2 \times S^1$, and if p-compatible ($p>0$), $\mathscr{K}=L(p, q)$ for some q.*

By direct calculation we have

COROLLARY 3. *If $\mathscr{K}=L(p, q)$, then q can be chosen to be $q=\det(S_1, \bar{S}_0)$.*

4. **Remarks.** Each closed orientable 3-manifold of genus 1 is a Seifert fibre space with base space S^2 and at most 2 singular fibres. In the model $\mathscr{K}$, the singular fibres can be taken to be $S_0'=0$ and $S_1'=0$ on the faces $z=0$ and $z=1$ respectively. Each fibring of $\mathscr{K}$ is then induced by a simple closed curve (fibre) $H=rx+sy$ which is nonhomotopic with S_0 and S_1 on the $z=1/2$ cross-section of $\mathscr{K}$, as these forms are self-cancelling under T_0 and T_1 respectively. Viewing $L(p, q)$ in the simplified form of Corollary 1, the singular fibre invariants of Seifert are readily obtained [**2**, p. 181].

A second perhaps pleasing visualization arises when considering the connected sum $M \# M'$ of two closed orientable 3-manifolds of genus 1, obtained by removing the interior of a 3-cell from each and then identifying the resulting boundaries via an orientation reversing homeomorphism. If $\mathscr{K}$ and $\mathscr{K}'$ are the models arising from T_0, T_1 and T_0', T_1' respectively, then $\mathscr{K} \# \mathscr{K}'$ can be obtained from $\mathscr{K}$ by removing the interior of a cube from its interior and then identifying the side faces of the resulting boundary straight across, the upper face via T_0' and the lower face via T_1'. (Reversing the orientation of $\mathscr{K}'$ is equivalent to interchanging the roles of T_0' and T_1'.) If $\mathscr{K}$ and $\mathscr{K}'$ are written in the simplified form of Corollary 1, we observe by performing the appropriate identifications on $\mathscr{K}'$, that when $\mathscr{K}'=S^2\times S^1$, then summing $\mathscr{K}'$ to $\mathscr{K}$ does correspond to the usual operation of adding a handle (removing the interiors of two disjoint 3-cells from $\mathscr{K}$ and then identifying the resulting boundaries by an orientation reversing homeomorphism). On the other hand, if $\mathscr{K}'=S^3$, then the "inner" boundary identifies to a point and $\mathscr{K} \# \mathscr{K}'=\mathscr{K}$, as is expected. Furthermore, the toroid decomposition of $M \# M'$ into two solid toroids of genus 2 is apparent by cutting $\mathscr{K} \# \mathscr{K}'$ with the plane $z=1/2$.

References

1. Harvey Cohn, *Sphere fibration induced by uniformization of modular group*, J. London Math. Soc. **43** (1968), 10–20. MR **37** #4015.

2. H. Seifert, *Topologie 3-dimensionaler gefaserter Räume*, Acta Math. **60** (1932), 147–238.

3. W. Threlfall and H. Seifert, *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen spharischen Räumen*, Math. Ann. **107** (1932), 543–586.

Department of Mathematics, University of Texas, El Paso, Texas 79968