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THE KERNEL OF A BLOCK OF A GROUP ALGEBRA

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ABSTRACT. Avoiding the theory of characters of finite groups and group algebras over fields of characteristic zero a ring theoretical proof is given for R. Brauer's theorem which asserts that the (modular) kernel of a block of a group algebra FG of a finite group over a field F of characteristic $p>0$ is a p-nilpotent normal subgroup of G.

If G is a finite group and if K is a splitting field for G which has characteristic $p>0$, then by R. Brauer [2] the intersection N of the kernels of the irreducible representations of the group G belonging to a block B of the group algebra KG is a p-nilpotent normal subgroup of G. Furthermore, if B is the principal block of KG, then N is the maximal p-nilpotent normal subgroup of G by Theorem 2 of R. Brauer [1]. Brauer's proofs use the theory of blocks of characters of G. In this note both results are proved for group algebras over *arbitrary* fields using only elementary ring theoretical methods.

Throughout this note FG denotes the group algebra of the finite group G over the field F with characteristic $p>0$, and $J(FG)$ is the Jacobson radical of FG. A block of a group algebra FG (in the sense of A. Rosenberg [6]) is a triple $B\leftrightarrow e\leftrightarrow\lambda$ consisting of a minimal direct (two-sided) summand B of FG, its identity element e, and a linear character λ of the center ZFG belonging to e. $B\leftrightarrow e\leftrightarrow\lambda$ is the principal block of FG, if e does not annihilate the irreducible FG-module belonging to the identity representation of G.

Concerning our terminology we refer to [3] and [5].

DEFINITION (CF. R. BRAUER [2, p. 494]). The *kernel* N_B *of the block* $B\leftrightarrow e\leftrightarrow\lambda$ of the group algebra FG is the intersection of the kernels of the irreducible (modular) representations of the finite group G belonging to B.

LEMMA 1. *The kernel* N_B *of the block* $B\leftrightarrow e\leftrightarrow\lambda$ *of the group algebra* FG *is the set*

$$N_B = \{g \in G \mid e(1-g) \in eJ(FG)\}.$$

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PROOF. Let $e=\sum_{i=1}^{s} e_i$ be a decomposition of e into orthogonal primitive idempotents of FG. Let $V_i=e_iFG/e_iJ(FG)$, $i=1,2,\cdots,r\leqq s$, be the set of all simple (not isomorphic) FG-modules belonging to the block $B\leftrightarrow e\leftrightarrow\lambda$. If f_i denotes the irreducible representation of G determined by V_i, and if $g\in N_B=\bigcap_{i=1}^{r}\ker f_i$, then $e_i(1-g)\in e_iJ(FG)$ for $i=1,2,\cdots,s$. Hence $N_B\leqq\{g\in G \mid e(1-g)\in eJ(FG)\}$.

Conversely, if $g\in G$ satisfies $e(1-g)\in eJ(FG)$, then $v_ig=v_ieg=v_i$ for every $v_i\in V_i$, $i=1,2,\cdots,r$, because $V_iJ(FG)=0$, and $v_i=v_ie$ for every $v_i\in V_i$ and $i=1,2,\cdots,r$. Thus Lemma 1 holds.

LEMMA 2. *Let $B\leftrightarrow e\leftrightarrow\lambda$ be a block of the group algebra FG. Then the following properties of $g\in G$ are equivalent:*

(1) $e=eg$,

(2) *g is a p-regular element of the kernel N_B.*

PROOF. Clearly every $g\in G$ satisfying (1) belongs to N_B. If g is not p-regular, then by Lemma 5.1 of [5] we may assume that g has order p. Let D be the cyclic subgroup generated by g, and let $e=\sum_{i=1}^{t}\lambda_ig_i$, $\lambda_i\in F$. Then $e=eg=ge$ implies that D acts fixed-point free on $\{g_i \mid i=1,2,\cdots,t\}$ by left multiplication. Thus we may assume that

$$e=\sum_{j=1}^{k}\lambda_j(g+g^2+\cdots+g^{p-1}+g^p)g_j,\quad\text{where } k=tp^{-1}.$$

As $e=e^2$ and $e=eg^s$ for $s=1,2,\cdots$, it follows that

$$e=e^2=\sum_{j=1}^{k}\lambda_j(eg+eg^2+\cdots+eg^{p-1}+eg^p)g_j=\sum_{j=1}^{k}p\lambda_jeg_j=0,$$

a contradiction. Thus (1) implies (2).

Conversely, if g is a p-regular element of N_B, then, by Lemma 1, $0=e(1-g)^{p^k}=e-eg^{p^k}$ for some integer $k\geqq 0$, because the Jacobson radical $J(FG)$ of FG is nilpotent. If r is the order of g, then $(p^k,r)=1$. Hence $g=g^{p^ku+rv}$ for integers u and v. Thus $e=eg^{p^k}$ implies

$$eg=eg^{p^ku+rv}=eg^{p^ku}=e^u=e,$$

which completes the proof of Lemma 2.

R. Brauer's results ([2, Proposition (3D)], and [1, Theorem 2]) are now easily derived from these two lemmas.

THEOREM 1. *The kernel N_B of the block $B\leftrightarrow e\leftrightarrow\lambda$ of the group algebra FG is a p-nilpotent normal subgroup of G.*

Moreover, if U is the maximal p-regular normal subgroup of N_B, then N_B/U is the maximal normal p-subgroup of G/U.

PROOF. Let U be the set of all p-regular elements of N_B. Then U is a normal subgroup of G by Lemma 2. As the Jacobson radical $J(FG)$ of FG is nilpotent, Lemma 1 at once implies that N_B/U is a normal p-subgroup of G/U. Thus N_B is p-nilpotent.

Since $(|U|,p)=1$ Theorem 2.1 of D. A. R. Wallace [7] asserts that $f=(1/|U|)\sum_{u\in U} u$ is a central idempotent of FG such that $fFG\cong F(G/U)$. As $e=ef=fe$ by Lemma 2, it now follows from Dickson's lemma (cf. [5, Lemma 4.3]) that the maximal normal p-subgroup S/U of G/U is N_B/U.

THEOREM 2. *The kernel N_B of the principal block $B\leftrightarrow e\leftrightarrow\lambda$ of the group algebra FG is the maximal p-nilpotent normal subgroup of the finite group G.*

PROOF. By Theorem 1 it suffices to show that the maximal p-regular normal subgroup Q of G is contained in N_B. As in the proof of Lemma 12.18 of Huppert [4] we consider the central idempotent $f=(1/|Q|)\sum_{q\in Q} q$ of FG. Clearly

$$f(1-q)=0 \quad \text{for every } q\in Q.$$

If φ is the identity representation of FG, and if φ' is its restriction to the center ZFG of FG, then φ' is a linear character of ZFG which is equivalent to λ. Thus we may identify λ with φ', and we obtain

$$1=\varphi(f)=\varphi'(f)=\lambda(ef).$$

Hence $e=ef$, which implies

$$e(1-q)=ef(1-q)=0 \quad \text{for every } q\in Q.$$

Thus $Q\leqq N_B$ by Lemma 1, which completes the proof of Theorem 2.

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