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# THE KERNEL OF A BLOCK OF A GROUP ALGEBRA

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ABSTRACT. Avoiding the theory of characters of finite groups and group algebras over fields of characteristic zero a ring theoretical proof is given for R. Brauer's theorem which asserts that the (modular) kernel of a block of a group algebra $FG$ of a finite group over a field $F$ of characteristic $p>0$ is a $p$-nilpotent normal subgroup of $G$.

If $G$ is a finite group and if $K$ is a splitting field for $G$ which has characteristic $p>0$, then by R. Brauer [2] the intersection $N$ of the kernels of the irreducible representations of the group $G$ belonging to a block $B$ of the group algebra $KG$ is a $p$-nilpotent normal subgroup of $G$. Furthermore, if $B$ is the principal block of $KG$, then $N$ is the maximal $p$-nilpotent normal subgroup of $G$ by Theorem 2 of R. Brauer [1]. Brauer's proofs use the theory of blocks of characters of $G$. In this note both results are proved for group algebras over *arbitrary* fields using only elementary ring theoretical methods.

Throughout this note $FG$ denotes the group algebra of the finite group $G$ over the field $F$ with characteristic $p>0$, and $J(FG)$ is the Jacobson radical of $FG$. A block of a group algebra $FG$ (in the sense of A. Rosenberg [6]) is a triple $B\leftrightarrow e\leftrightarrow\lambda$ consisting of a minimal direct (two-sided) summand $B$ of $FG$, its identity element $e$, and a linear character $\lambda$ of the center $ZFG$ belonging to $e$. $B\leftrightarrow e\leftrightarrow\lambda$ is the principal block of $FG$, if $e$ does not annihilate the irreducible $FG$-module belonging to the identity representation of $G$.

Concerning our terminology we refer to [3] and [5].

DEFINITION (CF. R. BRAUER [2, p. 494]). The *kernel* $N_B$ *of the block* $B\leftrightarrow e\leftrightarrow\lambda$ of the group algebra $FG$ is the intersection of the kernels of the irreducible (modular) representations of the finite group $G$ belonging to $B$.

LEMMA 1. *The kernel* $N_B$ *of the block* $B\leftrightarrow e\leftrightarrow\lambda$ *of the group algebra* $FG$ *is the set*

$$N_B = \{g \in G \mid e(1-g) \in eJ(FG)\}.$$

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PROOF. Let $e=\sum_{i=1}^{s} e_i$ be a decomposition of $e$ into orthogonal primitive idempotents of $FG$. Let $V_i=e_iFG/e_iJ(FG)$, $i=1, 2, \cdots, r\leqq s$, be the set of all simple (not isomorphic) $FG$-modules belonging to the block $B\leftrightarrow e\leftrightarrow\lambda$. If $f_i$ denotes the irreducible representation of $G$ determined by $V_i$, and if $g\in N_B=\bigcap_{i=1}^{r}\ker f_i$, then $e_i(1-g)\in e_iJ(FG)$ for $i=1, 2, \cdots, s$. Hence $N_B\leqq\{g\in G \mid e(1-g)\in eJ(FG)\}$.

Conversely, if $g\in G$ satisfies $e(1-g)\in eJ(FG)$, then $v_ig=v_ieg=v_i$ for every $v_i\in V_i$, $i=1, 2, \cdots, r$, because $V_iJ(FG)=0$, and $v_i=v_ie$ for every $v_i\in V_i$ and $i=1, 2, \cdots, r$. Thus Lemma 1 holds.

LEMMA 2. *Let $B\leftrightarrow e\leftrightarrow\lambda$ be a block of the group algebra FG. Then the following properties of $g\in G$ are equivalent*:

(1) $e=eg$,

(2) *g is a p-regular element of the kernel $N_B$.*

PROOF. Clearly every $g\in G$ satisfying (1) belongs to $N_B$. If $g$ is not $p$-regular, then by Lemma 5.1 of [5] we may assume that $g$ has order $p$. Let $D$ be the cyclic subgroup generated by $g$, and let $e=\sum_{i=1}^{t}\lambda_ig_i$, $\lambda_i\in F$. Then $e=eg=ge$ implies that $D$ acts fixed-point free on $\{g_i \mid i=1, 2, \cdots, t\}$ by left multiplication. Thus we may assume that

$$e=\sum_{j=1}^{k}\lambda_j(g+g^2+\cdots+g^{p-1}+g^p)g_j, \quad \text{where } k=tp^{-1}.$$

As $e=e^2$ and $e=eg^s$ for $s=1, 2, \cdots$, it follows that

$$e=e^2=\sum_{j=1}^{k}\lambda_j(eg+eg^2+\cdots+eg^{p-1}+eg^p)g_j=\sum_{j=1}^{k}p\lambda_jeg_j=0,$$

a contradiction. Thus (1) implies (2).

Conversely, if $g$ is a $p$-regular element of $N_B$, then, by Lemma 1, $0=e(1-g)^{p^k}=e-eg^{p^k}$ for some integer $k\geqq 0$, because the Jacobson radical $J(FG)$ of $FG$ is nilpotent. If $r$ is the order of $g$, then $(p^k, r)=1$. Hence $g=g^{p^ku+rv}$ for integers $u$ and $v$. Thus $e=eg^{p^k}$ implies

$$eg=eg^{p^ku+rv}=eg^{p^ku}=e^u=e,$$

which completes the proof of Lemma 2.

R. Brauer's results ([2, Proposition (3D)], and [1, Theorem 2]) are now easily derived from these two lemmas.

THEOREM 1. *The kernel $N_B$ of the block $B\leftrightarrow e\leftrightarrow\lambda$ of the group algebra FG is a p-nilpotent normal subgroup of G.*

*Moreover, if U is the maximal p-regular normal subgroup of $N_B$, then $N_B/U$ is the maximal normal p-subgroup of $G/U$.*

PROOF. Let $U$ be the set of all $p$-regular elements of $N_B$. Then $U$ is a normal subgroup of $G$ by Lemma 2. As the Jacobson radical $J(FG)$ of $FG$ is nilpotent, Lemma 1 at once implies that $N_B/U$ is a normal $p$-subgroup of $G/U$. Thus $N_B$ is $p$-nilpotent.

Since $(|U|,p)=1$ Theorem 2.1 of D. A. R. Wallace [7] asserts that $f=(1/|U|)\sum_{u\in U} u$ is a central idempotent of $FG$ such that $fFG\cong F(G/U)$. As $e=ef=fe$ by Lemma 2, it now follows from Dickson's lemma (cf. [5, Lemma 4.3]) that the maximal normal $p$-subgroup $S/U$ of $G/U$ is $N_B/U$.

THEOREM 2. *The kernel $N_B$ of the principal block $B\leftrightarrow e\leftrightarrow\lambda$ of the group algebra $FG$ is the maximal $p$-nilpotent normal subgroup of the finite group $G$.*

PROOF. By Theorem 1 it suffices to show that the maximal $p$-regular normal subgroup $Q$ of $G$ is contained in $N_B$. As in the proof of Lemma 12.18 of Huppert [4] we consider the central idempotent $f=(1/|Q|)\sum_{q\in Q} q$ of $FG$. Clearly

$$f(1-q)=0 \quad \text{for every } q\in Q.$$

If $\varphi$ is the identity representation of $FG$, and if $\varphi'$ is its restriction to the center $ZFG$ of $FG$, then $\varphi'$ is a linear character of $ZFG$ which is equivalent to $\lambda$. Thus we may identify $\lambda$ with $\varphi'$, and we obtain

$$1=\varphi(f)=\varphi'(f)=\lambda(ef).$$

Hence $e=ef$, which implies

$$e(1-q)=ef(1-q)=0 \quad \text{for every } q\in Q.$$

Thus $Q\leqq N_B$ by Lemma 1, which completes the proof of Theorem 2.

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