PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 1, January 1973

A NOTE ON HALL'S LEMMA

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ABSTRACT. Let H be the half plane $\{z:\operatorname{Re} z>0\}$. Let γ be a Jordan arc joining $z=0$ and $z=e^{i\alpha}$ $(0\leqq\alpha<\pi/2)$ in $H\cap\{|z|\leqq 1\}$. Let γ^* be the segment $z=iy$ $(0\leqq y\leqq 1)$ of the imaginary axis. If $\omega(z,\gamma)$ is the harmonic measure of γ with respect to $H\backslash\gamma$ and $\omega(z,\gamma^*)$ the harmonic measure of γ^* with respect to H, then $\omega(x+iy,\gamma)>\omega(x-i\,|y|,\gamma^*)$.

1. **Statement of result.** Let $H=\{z:\operatorname{Re} z>0\}$, let γ be a Jordan arc connecting $z=0$ to $z=e^{i\alpha}$ $(0\leqq\alpha<\pi/2)$ in $H\cap\{z:|z|\leqq 1\}$, and let $\gamma^*=\{z=iy:0\leqq y\leqq 1\}$ be the circular projection of γ onto the positive imaginary axis. Hall's lemma (see for example Duren [**1**, p. 208] or Fuchs [**2**, p. 82]) relates the harmonic measure $\omega(z,\gamma)$ of γ with respect to $H\backslash\gamma$ to the harmonic measure $\omega(z,\gamma^*)$ of γ^* with respect to H. We shall prove

THEOREM. *If $z=x+iy\in H$ and $\hat{z}=x-i|y|$, we have*

$$\omega(z,\gamma)>\omega(\hat{z},\gamma^*). \tag{1}$$

Hall's lemma admits a more general set $E\subset H$ instead of our arc γ but gives only $\omega(z,E)>\frac{2}{3}\omega(\hat{z},E^*)$. Our proof is in two parts in the first of which we allow more than one arc in H.

2. **First part of proof.** (a) Let $D=\{z:|z|<1\}$, γ a Jordan arc in D connecting $-a$ to $+a$ $(0<a<1)$, and $\gamma'=[-a,+a]$. Let u be harmonic in $D\backslash\gamma$ with boundary values $u=0$ on ∂D and $u=1$ on γ. *We claim that*

$$u(z)>\tfrac{1}{2}\quad \textit{for } z\in\gamma'. \tag{2}$$

To see this, let F be the two-sheeted Riemann surface above D with winding points $\pm a$; γ separates F into an upper sheet carrying $D\backslash\gamma$ and a lower sheet. Denote by u_1 the function harmonic on F with boundary values 0 and 1 on the upper and lower unit circle, respectively. Then $u=u_1=0$ on ∂D, $u>u_1$ on γ, and hence $u>u_1$ in $D\backslash\gamma$. Since $u_1=\frac{1}{2}$ on γ', we obtain (2).

Received by the editors June 5, 1972.
AMS (MOS) subject classifications (1970). Primary 30A44.
Key words and phrases. Harmonic measure, Hall lemma.

(b) Let γ_j $(j=1, 2, \cdots, n)$ be n disjoint Jordan arcs in H and let γ'_j be the circular arcs connecting the end points of γ_j in H and belonging to circles orthogonal to ∂H. Put $\Gamma=\bigcup \gamma_j$ and $\Gamma'=\bigcup \gamma'_j$ so that $H\backslash\Gamma$ is a region (of connectivity $n+1$) while $H\backslash\Gamma'$ is an open set. Let u be harmonic and bounded in $H\backslash\Gamma$ with boundary values $u=0$ on ∂H and $u=1$ on Γ; u_1 with boundary values 1 on Γ' is defined similarly. Together with their boundary values, u and u_1 are defined throughout H. *We now claim that*

$$u(z) \geqq \tfrac{1}{2}u_1(z) \quad \textit{for } z \in H. \tag{3}$$

To see this, consider any of the subregions, g, into which H is divided by $\Gamma\cup\Gamma'$. Clearly (3) is satisfied at those points of ∂g which belong to $\partial H\cup\Gamma$. If $z \in \partial g$ is on Γ', for example $z \in \gamma'_1$, consider the function $\tilde{u}$ harmonic and bounded in $H\backslash\gamma_1$ with boundary values $\tilde{u}=0$ on ∂H and $\tilde{u}=1$ on γ_1. Obviously $u\geqq\tilde{u}$ in H, and $\tilde{u}>\frac{1}{2}$ on γ'_1 follows from (2) after a suitable linear mapping. Thus (3) holds on ∂g and hence in g.

3. **Second part of proof.** We return to our original arc γ and let γ' be the circular arc connecting $z=0$ to $z=e^{i\alpha}$, orthogonal to ∂H at $z=0$. Application of (3) gives

$$\omega(z, \gamma) \geqq \tfrac{1}{2}\omega(z, \gamma') \quad \text{for } z \in H, \tag{4}$$

so that it remains to estimate $\omega(z, \gamma')$. We map H onto H by

$$w = \frac{1}{\cos\alpha}\left(\frac{1}{z} + i\sin\alpha\right)$$

which carries γ' onto $\gamma'_w=\{w:w\geqq 1\}$, γ^* onto $\gamma^*_w=\{w=iv:v\leqq -\mathrm{tg}(\beta/2)\}$ with $\alpha+\beta=\pi/2$, and z, $\hat{z}$ onto w, $\hat{w}$ which are symmetric with respect to $\{v:v=\mathrm{tg}\,\alpha\}$ if $\hat{z}=\bar{z}$.

Now the level curves of the harmonic measure of γ'_w with respect to $H\backslash\gamma'_w$ are hyperbolas

$$h_\delta = \left\{(u, v):\frac{u^2}{\cos^2\delta} - \frac{v^2}{\sin^2\delta} = 1\right\}$$

such that $\omega(w, \gamma'_w)=1-2\delta/\pi$ for $w \in h_\delta$ $(0<\delta<\pi/2)$; this is because $w=\frac{1}{2}(\sqrt{w_1}+1/\sqrt{w_1})$ maps $\{w_1:\mathrm{Im}\, w_1>0\}$ onto $H\backslash\gamma'_w$. On the other hand we have $2\omega(w, \gamma^*_w)=1-2\delta/\pi$ for $w \in l_\delta=\{(u, v):v=u\cdot\mathrm{tg}\,\delta-\mathrm{tg}(\beta/2)\}$ $(-\pi/2<\delta<\pi/2)$; note that, for $\delta>0$, l_δ meets h_δ exactly once in (u_0, v_0) with $v_0<\mathrm{tg}\,\alpha$.

Therefore, for any w with $\mathrm{Im}\, w\geqq\mathrm{tg}\,\alpha$, lying on h_δ for some $\delta \in (0, \pi/2)$, we have

$$\omega(w, \gamma'_w) = 1 - 2\delta/\pi > 2\omega(w, \gamma^*_w) = 2\omega(\hat{w}, \gamma^*_w).$$

If $0 \leqq \text{Im } w < \text{tg } \alpha$, then $\omega(w, \gamma'_w) > \omega(\hat{w}, \gamma'_w) > 2\omega(\hat{w}, \gamma^*_w)$ by what we just proved, and if $\text{Im } w < 0$ we get

$$\omega(w, \gamma'_w) = \omega(\bar{w}, \gamma'_w) > 2\omega(\hat{\bar{w}}, \gamma^*_w) > 2\omega(\hat{w}, \gamma^*_w).$$

In all cases $\omega(w, \gamma'_w) > 2\omega(\hat{w}, \gamma^*_w)$, i.e. $\omega(z, \gamma') > 2\omega(\hat{z}, \gamma^*)$ which proves our assertion.

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