A NOTE ON HALL'S LEMMA

DIETER GAIER

Abstract. Let $H$ be the half plane $\{z: \mathfrak{Re} z > 0\}$. Let $\gamma$ be a Jordan arc joining $z=0$ and $z=e^{i\alpha}$ ($0 \leq \alpha < \pi/2$) in $H \cap \{z: |z| \leq 1\}$. Let $\gamma^*$ be the segment $z=iy$ ($0 \leq y \leq 1$) of the imaginary axis. If $\omega(z, \gamma)$ is the harmonic measure of $\gamma$ with respect to $H \backslash \gamma$ and $\omega(z, \gamma^*)$ the harmonic measure of $\gamma^*$ with respect to $H$, then $\omega(z+iy, \gamma) > \omega(z-i|y|, \gamma^*)$.

1. Statement of result. Let $H=\{z: \mathfrak{Re} z > 0\}$, let $\gamma$ be a Jordan arc connecting $z=0$ to $z=e^{i\alpha}$ ($0 \leq \alpha < \pi/2$) in $H \cap \{z: |z| \leq 1\}$, and let $\gamma^* = \{z=iy: 0 \leq y \leq 1\}$ be the circular projection of $\gamma$ onto the positive imaginary axis. Hall’s lemma (see for example Duren [1, p. 208] or Fuchs [2, p. 82]) relates the harmonic measure $\omega(z, \gamma)$ of $\gamma$ with respect to $H \backslash \gamma$ to the harmonic measure $\omega(z, \gamma^*)$ of $\gamma^*$ with respect to $H$. We shall prove

**Theorem.** If $z=x+iy \in H$ and $\bar{z}=x-i|y|$, we have

$$\omega(z, \gamma) > \omega(z, \gamma^*).$$

Hall’s lemma admits a more general set $E \subset H$ instead of our arc $\gamma$ but gives only $\omega(z, E) > \omega(z, E^*)$. Our proof is in two parts in the first of which we allow more than one arc in $H$.

2. First part of proof. (a) Let $D=\{z: |z|<1\}$, $\gamma$ a Jordan arc in $D$ connecting $-a$ to $+a$ ($0 < a < 1$), and $\gamma' = [-a, +a]$. Let $u$ be harmonic in $D \backslash \gamma$ with boundary values $u=0$ on $\partial D$ and $u=1$ on $\gamma$. We claim that

$$u(z) > \tfrac{1}{2} \quad \text{for } z \in \gamma'.$$

To see this, let $F$ be the two-sheeted Riemann surface above $D$ with winding points $\pm a$; $\gamma$ separates $F$ into an upper sheet carrying $D \backslash \gamma$ and a lower sheet. Denote by $u_1$ the function harmonic on $F$ with boundary values 0 and 1 on the upper and lower unit circle, respectively. Then $u=u_1=0$ on $\partial D$, $u > u_1$ on $\gamma$, and hence $u > u_1$ in $D \backslash \gamma$. Since $u_1 = \tfrac{1}{2}$ on $\gamma'$, we obtain (2).

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(b) Let \( \gamma_j \) (\( j = 1, 2, \cdots, n \)) be \( n \) disjoint Jordan arcs in \( H \) and let \( \gamma_j' \) be the circular arcs connecting the end points of \( \gamma_j \) in \( H \) and belonging to circles orthogonal to \( \partial H \). Put \( \Gamma = \bigcup \gamma_j \) and \( \Gamma' = \bigcup \gamma_j' \) so that \( H \setminus \Gamma \) is a region (of connectivity \( n+1 \)) while \( H \setminus \Gamma' \) is an open set. Let \( u \) be harmonic and bounded in \( H \setminus \Gamma \) with boundary values \( u = 0 \) on \( \partial H \) and \( u = 1 \) on \( \Gamma \); \( u_j \) with boundary values 1 on \( \Gamma_j \) is defined similarly. Together with their boundary values, \( u \) and \( u_j \) are defined throughout \( H \). We now claim that

\[
(3) \quad u(z) \geq \frac{1}{n} u_j(z) \quad \text{for} \quad z \in H.
\]

To see this, consider any of the subregions, \( g \), into which \( H \) is divided by \( \Gamma \cup \Gamma' \). Clearly (3) is satisfied at those points of \( \partial y \) which belong to \( \partial H \cup \Gamma \). If \( z \in \partial g \) is on \( \Gamma' \), for example \( z \in \gamma_j' \), consider the function \( \tilde{u} \) harmonic and bounded in \( H \setminus \gamma_j' \), with boundary values \( \tilde{u} = 0 \) on \( \partial H \) and \( \tilde{u} = 1 \) on \( \gamma_j' \). Obviously \( \tilde{u} \geq u \) in \( H \setminus \gamma_j' \), and \( u \geq \frac{1}{n} \) on \( \gamma_j' \) follows from (2) after a suitable linear mapping. Thus (3) holds on \( \partial g \) and hence in \( g \).

3. Second part of proof. We return to our original arc \( \gamma \) and let \( \gamma' \) be the circular arc connecting \( z = 0 \) to \( z = e^{it} \), orthogonal to \( \partial H \) at \( z = 0 \). Application of (3) gives

\[
(4) \quad \omega(z, \gamma) \geq \frac{1}{2} \omega(z, \gamma') \quad \text{for} \quad z \in H,
\]

so that it remains to estimate \( \omega(z, \gamma') \). We map \( H \) onto \( H \) by

\[
w = \frac{1}{\cos \alpha} \left( \frac{1}{z} + i \sin \alpha \right)
\]

which carries \( \gamma' \) onto \( \gamma'_w = \{ w : w \geq 1 \} \), \( \gamma^* \) onto \( \gamma^*_w = \{ w = i v : v \leq -\tan(\beta/2) \} \) with \( \alpha + \beta = \pi/2 \), and \( z, \tilde{z} \) onto \( w, \tilde{w} \) which are symmetric with respect to \( \{ v : v = \tan \alpha \} \) if \( \tilde{z} = \bar{z} \).

Now the level curves of the harmonic measure of \( \gamma'_w \) with respect to \( H \setminus \gamma'_w \) are hyperbolas

\[
h_\delta = \left\{ (u, v) : \frac{u^2}{\cos^2 \delta} - \frac{v^2}{\sin^2 \delta} = 1 \right\}
\]
such that \( \omega(w, \gamma'_w) = 1 - 2\delta/\pi \) for \( w \in h_\delta \) \((0 < \delta < \pi/2)\); this is because \( w = \frac{1}{\sqrt{2}}(\sqrt{w_1} + 1/\sqrt{w_1}) \) maps \( \{ w_1 : \text{Im} w_1 > 0 \} \) onto \( H \setminus \gamma'_w \). On the other hand we have \( 2\omega(w, \gamma^*_w) = 1 - 2\delta/\pi \) for \( w \in l_\delta = \{ (u, v) : v = u \cdot \tan \delta - \tan(\beta/2) \} \) \((-\pi/2 < \delta < \pi/2)\); note that, for \( \delta > 0 \), \( l_\delta \) meets \( h_\delta \) exactly once in \( (u_\delta, v_\delta) \) with \( v_\delta < \tan \alpha \).

Therefore, for any \( w \) with \( \text{Im} w \geq \tan \alpha \), lying on \( h_\delta \) for some \( \delta \in (0, \pi/2) \), we have

\[
\omega(w, \gamma'_w) = 1 - 2\delta/\pi > 2\omega(w, \gamma^*_w) = 2\omega(\tilde{w}, \gamma^*_w).
\]
If $0 \leq \text{Im } w < \tan \alpha$, then $\omega(w, \gamma'_w) > \omega(\hat{w}, \gamma'_w) > 2\omega(\hat{w}, \gamma^*_w)$ by what we just proved, and if $\text{Im } w < 0$ we get

$$\omega(w, \gamma'_w) = \omega(\hat{w}, \gamma'_w) > 2\omega(\hat{w}, \gamma^*_w) > 2\omega(\hat{w}, \gamma^*_w).$$

In all cases $\omega(w, \gamma'_w) > 2\omega(\hat{w}, \gamma^*_w)$, i.e. $\omega(z, \gamma') > 2\omega(\hat{z}, \gamma^*)$ which proves our assertion.

**References**


**Department of Mathematics, Universität Gießen, Gießen, Federal Republic of Germany**