PD-MINIMAL SOLUTIONS OF $\Delta u = Pu$ ON OPEN RIEHMANN SURFACES

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Abstract. By means of the Royden compactification of an open Riemann surface $R$ necessary and sufficient conditions are given for a Dirichlet-finite solution of $\Delta u = Pu \ (P \geq 0, \ P \neq 0)$ to be $PD$-minimal on $R$. A relation between $PD$-minimal solutions and $HD$-minimal solutions is obtained. In addition it is shown that the dimension of the space of $PD$-solutions is the same as the number of $P$-energy nondensity points in the finite dimensional case.

Let $P(z) \, dx \, dy \ (z = x + iy), \ P \neq 0$, be a nonnegative $C^1$ differential on an open Riemann surface $R$. Denote by $PD(R)$ the Hilbert space of all Dirichlet-finite solutions of the second-order selfadjoint elliptic partial differential equation

$$\Delta u(z) = P(z)u(z)$$

on $R$ where $\Delta u(z) \, dx \, dy = d^* \, du(z)$. The scalar product is given by $(u, v) = D_R(u, v) = \int_R du \wedge \ast dv$, not the energy integral $E_R(u, v) = D_R(u, v) + \int_R P^2 uv$. Observe that the only constant solution of $\Delta u = Pu$ is the identically zero solution. The classification problem with respect to $\Delta u = Pu$ was initiated by Ozawa [9] who investigated the class $PE(R)$ of energy-finite solutions of (1) on $R$. The class $PD(R)$ itself was first considered by Royden [10] in 1959. A little later the works of Nakai ([5], [6]) gave impetus to the theory of the class $PD(R)$. Recent contributions to the study of $PD(R)$ are contained in papers by Nakai ([7], [8]), Glasner-Katz [2], and Singer ([12], [13]).

The energy integral $E_R(u) = E_R(u, u)$ plays the same role as the Dirichlet integral $D_R(u) = D_R(u, u)$ in the harmonic case, i.e. solutions of $\Delta u = 0$, and the class $PE(R)$ likewise shares many properties possessed by the class $HD(R)$ of Dirichlet-finite harmonic functions (see, for example, Ozawa [9], Glasner-Katz [2], Kwon-Sario-Schiff ([3], [4])). However, the class $PD(R)$ is quite different in nature from $HD(R)$. Nevertheless it does share...
some common properties with \(HD(R)\). For example, Nakai [8] has shown recently that the Virtanen identity \(O_{\text{HD}}=O_{\text{HBD}}\) is also valid for \(PD(R)\); namely, \(O_{\text{PD}}=O_{\text{PD}}\), where \(PB(D(R))\) is the class of bounded \(PD\)-functions on \(R\).

The purpose of this paper is to give a necessary and sufficient condition for a \(PD\)-function to be \(PD\)-minimal. Although the statement itself is similar to that for \(HD\)-functions new techniques are required for the proofs. The most important tool used is the Royden harmonic boundary and in particular the subset \(\Delta_p\) of \(P\)-energy nondensity points introduced by Nakai [7]. Further we give a relationship between \(HD\)-minimality and \(PD\)-minimality. Finally we also state a relation between the cardinality of \(\Delta_p\) and the dimension of \(PD(R)\) whenever the latter is finite. For the reader’s convenience we shall briefly review some preliminaries in §1.

1. Let \(R^\ast\) be the Royden compactification of \(R\) (for details see e.g. Sario-Nakai [11]). Denote by \(\Gamma = R^\ast - R\) the Royden boundary of \(R\) and by \(\Delta = \Delta_R\) the Royden harmonic boundary of \(R\), consisting of points of \(\Gamma\) which are regular for the harmonic Dirichlet problem. A point \(Z^\ast\) in \(\Delta\) will be called a \(P\)-energy nondensity point (cf. [7]) if there exists an open neighborhood \(U^\ast\) of \(Z^\ast\) in \(R^\ast\) such that

\[
\int_{U \times U} G_{U}(z, \zeta)P(z)P(\xi)\, dx\, dy\, d\xi\, d\eta < \infty \quad (\zeta = \xi + i\eta)
\]

for \(z \in U\), where \(U = U^\ast \cap R\) and \(G_{U}\) is the harmonic Green’s function of \(U\).

2. If \(R\) is parabolic then \(PD(R) = \{0\}\) (cf. Royden [10]). We therefore assume throughout the paper that \(R\) is hyperbolic. Denote by \(\hat{M}(R)\) the class of all Dirichlet-finite Tonelli functions on \(R\) and by \(\hat{M}_\Delta(R)\) the subclass of \(\hat{M}(R)\) consisting of functions \(g\) such that \(g|_{\Delta} = 0\). We then have the orthogonal decomposition (cf. [11]):

\[
\hat{M}(R) = HD(R) + \hat{M}_\Delta(R).
\]

The subset \(M(R)\) consisting of all bounded members of \(\hat{M}(R)\) is called the Royden algebra of \(R\). It is known that \(M(R)\) is closed under the lattice operations \(f \cup g = \max(f, g)\), and \(f \cap g = \min(f, g)\). Moreover \(M(R)\) has the orthogonal decomposition

\[
M(R) = HB(D(R) + M_{\Delta}(R),
\]

where \(HB(D(R)\) is the class of bounded harmonic functions on \(R\) and \(M_{\Delta}(R)\) the subclass of \(M(R)\) consisting of functions \(g\) with \(g|_{\Delta} = 0\).

For each \(f \in \hat{M}(R)\) we denote by \(\Pi_R f \in HD(R)\) the harmonic projection of \(f\) on \(R\) characterized by \(f - \Pi_R f \in \hat{M}_\Delta(R)\). Since \(PD(R) \subseteq \hat{M}(R)\) we may
define the operator

\[ \Pi_R \mid PD(R) : PD(R) \to HD(R) \]

which is a vector space isomorphism from \( PD(R) \) onto \( \Pi_R(PD(R)) \) such that \( u > 0 \) is equivalent to \( \Pi_R u > 0 \) and

\[ \sup_R |u| = \sup_R |\Pi_R u| \quad (\text{cf. } [12]). \]

Moreover it can be shown that if \( u \in PD(R) \) then

\[ u = \Pi_R u + T_R u \]

where

\[ T_R u = -\frac{1}{2\pi} \int_R G_R(\cdot, \zeta)P(\zeta)u(\zeta) \, d\zeta \, d\eta \quad (\zeta = \xi + i\eta) \]

and also

\[ D_R(u) = D_R(\Pi_R u) + \frac{1}{2\pi} \int_{R \times R} G_R(z, \zeta)u(z)u(\zeta)P(z)P(\zeta) \, dx \, dy \, d\xi \, d\eta \]

(cf. [8]). If \( \Omega \) is an open subset of \( R \) with smooth relative boundary \( \partial \Omega \) (which may be empty in case \( \Omega = R \)) then for \( u \in PD(\Omega) \) we obtain representations for \( u \) and \( D_{\Omega}(u) \) as in (5), (6). Moreover

\[ T_{\Omega} u \mid (\partial \Omega) \cup (\bar{\Omega} \cap \Delta) = 0, \]

where \( \bar{\Omega} \) is the closure of \( \Omega \) in \( R^* \).

The following is an immediate consequence of the maximum principle for \( PD(R) \) (cf. Glasner-Katz [2]):

**Lemma 1.** If \( u \in PD(R) \) and \( u|\Delta = 0 \) then \( u \equiv 0 \).

3. Recall that \( \Delta_p \) is the set of \( P \)-energy nondensity points of \( \Delta \). Now we state (cf. Nakai [7]):

**Lemma 2.** If \( u \in PD(R) \) then \( u|\Delta - \Delta_p = 0 \).

**Proof.** Let \( Z_0 \in \Delta - \Delta_p \). Then for each neighborhood \( U^* \) of \( Z_0 \) in \( R^* \),

\[ \int_{U^* \times U} G_U(z, \zeta)P(z)P(\zeta) \, dx \, dy \, d\xi \, d\eta = \infty, \]

\( U = U^* \cap R \). Suppose to the contrary that \( u(Z_0) \neq 0 \). Since each \( u \in PD(R) \) possesses a Riesz decomposition (cf. [8]) as the difference of two non-negative \( PD \)-functions on \( R \) we may assume that \( \bar{u} \geq 0 \) and \( u(Z_0) > 0 \). Since \( u \) is continuous at \( z_0 \) there exists a neighborhood \( U^* \) of \( Z_0 \) in \( R^* \) such that
\( u \geq \delta > 0 \) in \( U^* \). But from (6) and the fact that \( D_U(u) \leq D_R(u) < \infty \) \((U = U^* \cap R)\) we have
\[
D_U(u) = D_U(\Pi_U u) + \frac{1}{2\pi} \int_{U \times U} G_U(z, \zeta) u(z) u(\zeta) P(z) P(\zeta) < \infty,
\]
which is impossible. Hence \( u(Z_0) = 0 \) as asserted.

**Corollary 1.** If \( u \in PD(R) \) and \( u|_{\Delta_p} = 0 \) then \( u \equiv 0 \).

The proof follows immediately from Lemmas 1 and 2.

4. A positive \( PD \)-function \( u \) on \( R \) which is not identically zero will be called \( PD \)-minimal if for any \( v \in PD(R) \) such that \( 0 \leq v \leq u \) there exists a constant \( c_v \) such that \( v = c_v u \) on \( R \) (for \( HD \)-minimal functions see Sario-Nakai [11]).

In contrast to \( HD \)-minimality which is characterized in terms of the entire harmonic boundary \( \Delta \), \( PD \)-minimality is stated solely in terms of \( \Delta_p \) as follows:

**Theorem 1.** A \( PD \)-function on \( R \) is \( PD \)-minimal if and only if there exists an isolated point \( Z_0 \in \Delta_p \) such that \( 0 < u(Z_0) \) and \( u = 0 \) on \( \Delta_p - \{Z_0\} \).

**Proof.** We first establish the sufficiency. Since \( \Delta = \Delta_p \cup (\Delta - \Delta_p) \) it follows from the hypothesis and Lemma 2 that \( u|\Delta - \{Z_0\} = 0 \). Now \( \Pi_R u \in HD(R) \) by (3) and from (7) we deduce that \( \Pi_R u(Z_0) = u(Z_0) > 0 \) and \( \Pi_R u = 0 \) on \( \Delta - \{Z_0\} \). Hence \( \Pi_R u \) is \( HD \)-minimal, and in particular strictly positive and bounded (cf. [11]). From (4) it follows that \( u \) is bounded. For any \( v \in PD(R) \) with \( 0 \leq v \leq u \) on \( R \) it follows from the continuity of \( PD \)-functions on \( \Delta \) that \( v = 0 \) on \( \Delta - \{Z_0\} \) and \( 0 \leq v(Z_0) < \infty \). Hence \( c_v u - v = 0 \) on \( \Delta \) where \( c_v = v(Z_0)/u(Z_0) \). By Lemma 1, \( v = c_v u \) on \( R \) and \( u \) is \( PD \)-minimal as was to be shown.

Conversely, assume that \( u \) is \( PD \)-minimal. Since \( u \neq 0 \) by Corollary 1 there exists a point \( Z_0 \in \Delta_p \) such that \( u(Z_0) > 0 \). There exists a neighborhood \( U^* \) of \( Z_0 \) as in (2). Suppose \( Z_0 \) is not an isolated point of \( \Delta_p \). Then consider any \( Z_1 \in \Delta_p \cap U^* \) with \( Z_1 \neq Z_0 \). We claim that \( u(Z_1) = 0 \). Suppose to the contrary that \( u(Z_1) > 0 \). Note that we may assume that \( \partial U \) \((U = U^* \cap R)\) is smooth to begin with since we may modify \( U \) suitably otherwise. Select an \( f \in M(U) \) such that \( f(Z_0) = 1 \), \( f(Z_1) = 0 \), \( f|\partial U = 0 \), and \( 0 \leq f \leq 1 \) on \( U^* \). Here \( M(U) \) is the Royden algebra of bounded Dirichlet-finite Tonelli functions on \( U \). Then \( h = \Pi_U (f \wedge u) \in HBD(U) \), \( 0 \leq h \leq u \) on \( U^* \), \( h|\partial U = 0 \), \( h(Z_0) = 0 \), and \( h(Z_0) = (f \wedge u)Z_0 \). Using the approach of Nakai [7] we now construct an appropriate \( w \in PBD(R) \). We sketch the procedure here for the sake of completeness. By the method of exhaustion it is seen that the integral equation of the Fredholm type \((I - T_U) v = h\) has a unique solution \( v \in PD(U) \), where \( I \) is the identity operator. Now \( v|\partial U = 0 \), \( v(Z_0) = h(Z_0) \).
\( v(Z) = 0 \), and \( 0 \leq v \leq h \leq 1 \). \( v \) is a Dirichlet-finite subsolution of (1). By the exhaustion method again, and by the weak Dirichlet principle (cf. [8]) we obtain a \( w \in PBD(R) \) such that \( v \leq w \leq 1 \). Now \( w|\Delta \cap U^* = u|\Delta \cap U^* \) by construction and \( w = 0 \) on \( \Delta \cap (R^* - U^*) \). Therefore \( 0 \leq w \leq u \) on \( \Delta \) and hence on \( R \). It follows that there is a constant \( c_w \) such that \( w = c_w u \). But \( w(Z) = 0 = c_w u(Z) > 0 \), a contradiction. Hence \( u(Z) = 0 \) as asserted.

Since \( u \) is continuous at \( Z_0 \) and \( u(Z) = 0 \) for any \( Z \neq Z_0 \in U^* \cap \Delta_p \) it follows that \( Z_0 \) is an isolated point of \( \Delta_p \).

To complete the proof we now show \( u|\Delta_p - \{Z_0\} = 0 \). Observe that for the function \( w \in PBD(R) \) constructed above, \( w(Z_0) = (\mathfrak{f} \cap u)Z_0 \), \( w|\Delta_p - \{Z_0\} = 0 \) and \( 0 \leq w \leq u \) on \( \Delta_p \). Therefore \( w = c_w u \) on \( R \) and so if there exists a \( Z \in \Delta_p - \{Z_0\} \) such that \( u(Z) > 0 \) we obtain a contradiction \( w(Z) = 0 = c_w u(Z) > 0 \).

This completes the proof.

**Corollary 2.** If \( Z \in \Delta_p \) is isolated in \( \Delta_p \) then there always exists a \( u \in PBD(R) \) such that \( u(Z) > 0 \) and \( u = 0 \) on \( \Delta - \{Z\} \). Also any PD-function \( v \) on \( R \) has a finite value at \( Z \).

For a proof of the second part we may assume \( u \geq 0 \) on \( R \) since \( u \) has a Riesz decomposition. If \( v(Z) = \infty \) then for \( n = 1, 2, \ldots \) the inequality \( nu \leq v \) holds on \( \Delta \) and hence on \( R \). But this yields the contradiction \( v = \infty \).

5. A relation between PD-minimality and HD-minimality is given by

**Theorem 2.** If \( \Pi_R \) maps \( PD(R) \) onto \( HD(R) \) then \( u \in PD(R) \) is PD-minimal if and only if \( \Pi_R u \in HD(R) \) is HD-minimal.

**Proof.** First assume \( u \in PD(R) \) is PD-minimal. Then for any \( h \in HD(R) \) with \( 0 \leq h \leq \Pi_R u \) on \( R \) there exists a \( v \in PD(R) \) such that \( \Pi_R v = h \). From (5) and (7) we see that \( u = \Pi_R u \) and \( v = \Pi_R v \) on \( \Delta \). Hence \( 0 \leq v \leq u \) on \( \Delta \) and so there exists a constant \( c_v \) such that \( v = c_v u \) on \( R \). So \( h = \Pi_R v = c_v \Pi_R u \) as was to be shown. The converse follows similarly since \( \Pi_R \) is one-to-one.

6. In case \( 0 \leq \dim PD(R) < \infty \) we have the following PD-function analogue corresponding to that for HD-functions (cf. [11]) and for PE-functions (cf. [2]):

**Theorem 3.** \( \Delta_p \) contains exactly \( m \) points if and only if \( \dim PD = \dim PBD = m \).

**Proof.** First of all if \( m = 0 \), i.e. \( \Delta_p = \emptyset \) then any \( u \in PD(R) \) vanishes on \( \Delta \) by Lemma 2 and consequently \( u \equiv 0 \), i.e. \( \dim PD = \dim PBD = 0 \). Assume next that there are exactly \( m \geq 1 \) points \( Z_1, Z_2, \ldots, Z_m \in \Delta_p \).
Take neighborhoods $U_i$ of $Z_i$ such that $\bar{U}_i \cap \bar{U}_j = \emptyset$ ($i \neq j$) in $\mathbb{R}^*$. Modify (if necessary) each $U_i$ so that $\partial U_i$ is smooth. Choose $h_i \in HBD(U_i)$ such that $h_i|_{\partial U_i} = 0$, $0 \leq h_i \leq 1$ on $U_i$, and $h_i(Z_i) = 1$. As in the proof of Theorem 1 construct functions $u_i \in PD(U_i)$ such that $u_i|_{\partial U_i} = 0$, $u_i(Z_i) = 1$, and $0 \leq u_i \leq h_i \leq 1$ in $\bar{U}_i$. Setting $u_i = 0$ on $R - U_i$, we in turn construct as before $v_i \in PBD(R)$ such that $u_i \leq v_i \leq 1$ on $R$. For a given $v_i$ observe that $v_i(Z_i) = 0$ for $j \neq i$ since the $Z_i$ are regular points for the Dirichlet problem. It follows that the $v_i$, $i = 1, 2, \ldots, m$, are linearly independent in $PBD(R)$ and so $\dim PD(R) \geq \dim PBD(R) = m$.

Next let $w \in PD(R)$. Then $w$ has a Riesz decomposition $w = w_1 - w_2$, with $w_i \in PD(R)$, $w_i \geq 0$ on $R$. We claim that $w_i(Z_i) < \infty$, $j = 1, \ldots, m$. If not, say $w_i(Z_i) = \infty$; then for $c > 0$, $w_i - cw_j |_{\Delta} \geq 0$ and so $w_i \geq cw_j$ on $R$. But this implies $w_i(Z) = \infty$ for $Z \in R$, a contradiction. Since $w_1 |_{\Delta - \Delta_p} = 0$ it follows that $w = \sum_{i=1}^{m} (w_i(Z_i) - w_2(Z_i))v_i$ on $\Delta$ and hence on $R$. Therefore $\dim PD(\Delta) = \dim PBD(\Delta) = m$.

Conversely if $\dim PD = \dim PBD = m$ then $\Delta_p$ cannot contain more than $m$ points. For if there exist at least $m + 1$ points $Z_1, Z_2, \ldots, Z_{m+1} \in \Delta_p$ then as in the first part of the proof construct $m + 1$ linearly independent functions $v_1, v_2, \ldots, v_{m+1} \in PBD$, thereby contradicting $\dim PBD = m$. Hence $\Delta_p$ has $n$ points $0 \leq n \leq m$. As earlier in the proof there exist $n$ functions $v_i \in PBD(R)$ such that any $w \in PD(R)$ is a linear combination of these $v_i$. We conclude $\Delta_p$ has precisely $m$ points; and this completes the proof.

**Added in proof.** Results similar to those in this paper have been obtained by J. L. Schiff (A note on the space of Dirichlet-finite solutions of $\Delta u = Pu$ on a Riemann surface, Hiroshima Math. J. 2 (1972) (to appear)).

**References**


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